Theorem 2. *Let $r \geq 2$. If $p_1, p_2, \dots, p_r$ are the first r primes of the form $p_i = 4k_i + 3$, then the interval $(p_r, \prod_1^r p_i)$ contains at least $\lfloor \log_2(4(r-1)) \rfloor$ primes congruent to 3 modulo 4.*

Proof. For $r = 2$ or 3, the result can be checked by hand, so we assume that $r \geq 4$. Since $p_4 = 19 > 4 \cdot 4$, we have $4r < p_r$. If $1 \leq j \leq \lfloor \log_2(4r) \rfloor + 1$, it follows that $2^{j+1} < 4p_r$. Hence

$$\prod_1^r p_i - 2^{j+1} > 7p_r - 4p_r > p_r.$$

If r is odd, set $n_k = \prod_1^r p_i - 2^{k+1}$. Then $n_k \equiv 3(4)$ (since $3^2 \equiv 1(4)$). But no p_i divides n_k for $1 \leq i \leq r$, so the integer n_k has some prime factor $q_k \equiv 3(4)$ with $q_k > p_r$. If $j \neq k$, say $j > k$, the assumption that q_k also divides n_j leads to the same contradiction as earlier: since $n_k - n_j = 2^{j+1} - 2^{k+1} = 2^{k+1}(2^{j-k} - 1)$, we have $q_k \mid 2^{j-k} - 1$ and hence $q_k < p_r$. Thus, there are at least $\lfloor \log_2(4r) \rfloor + 1$ distinct primes of the form $4\ell + 3$ in $(p_r, \prod_1^r p_i)$. If r is even, the same argument applied to $r - 1$ shows that there are at least $\lfloor \log_2(4(r-1)) \rfloor + 1$ distinct primes of the form $4\ell + 3$ in $(p_{r-1}, \prod_1^{r-1} p_i)$. Since the first of these is p_r, there are at least $\lfloor \log_2(4(r-1)) \rfloor$ distinct primes of the form $4\ell + 3$ in $(p_r, \prod_1^{r-1} p_i)$, a subinterval of $(p_r, \prod_1^r p_i)$. ■

Universidad de La Rioja, 26004 Logroño, La Rioja, Spain.
aldaz@dmc.unirioja.es

Departament of Mathematics, University of Michigan, Ann Arbor MI 48109-1109
anabz@math.lsa.umich.edu

Smooth Interpolation, Hölder Continuity, and the Takagi–van der Waerden Function

Jack B. Brown and George Kozlowski

1. INTRODUCTION. Consider the classes of functions $f : [0, 1] \to \mathbb{R}$ indicated in the following diagram (where $0 < \beta < \alpha < 1$):

$$C^1 \subset Lip^1 \subset \bigcap_{0<\gamma<1} Lip^\gamma \subset Lip^\alpha \subset Lip^\beta \subset C,$$

in which C denotes the class of continuous functions, C^1 the class of continuously differentiable functions, and

$$Lip^\alpha = \{f \in C : \exists K > 0 \ni |f(x) - f(y)| < K|x - y|^\alpha \text{ for } x, y \in [0, 1]\}$$

the class of Lipschitz (or Hölder continuous) functions of order α. The symbol λ signifies Lebesgue measure. Marcinkiewicz [**7**] showed that there is a strong interpolation link between the classes C^1 and Lip^1 when he proved: *if f belongs to Lip^1, then for every $\epsilon > 0$ there exists g in C^1 such that $\lambda(\{x : f(x) \neq g(x)\}) < \epsilon$.* Marcinkiewicz actually required only that f be "pointwise" Lip^1 (i.e., that $f(x+t) = f(x) + O(t)$ at each x) and obtained similar results for higher order smoothness. Federer [**5**, p. 442] obtained the analogous Lip^1-C^1 result in higher dimensions, and Whitney [**13**] extended Federer's results to higher order smoothness (see also [**6**, p. 227]). Whitney described an example of a function ϕ of one variable that was designed to show that one could not weaken the requirement of f being in Lip^1 in the Marcinkiewicz theorem. He stated the following [**13**, p. 144]:

> Let $\phi_0(t)$ be the distance from t to the nearest integer. Using any sufficiently large integer a, set
>
> $$\phi_i(t) = 2^i \phi_0(a^i t)/a^i, \quad \phi(t) = \sum_{i=0}^{\infty} \phi_i(t).$$
>
> Then ϕ satisfies a Lipschitz condition of order $1-\alpha$, for any $\alpha > 0$; but [the conclusion of Marcinkiewicz's theorem] is not true for it.

The function ϕ_0 is the familiar "rooftop" function, with "roofs" having slopes ± 1. For $n > 0$ the roofs of ϕ_n have slopes $\pm 2^n$. The graphs of ϕ_1 and ϕ_2 on [0,1], with $a = 3$, are displayed in Figure 1. Whitney's statement would seem to suggest that for sufficiently large a in $\mathbb{N}$ the resulting function ϕ would belong to $\bigcap_{0<\gamma<1} Lip^\gamma$. Such, however, is not the case.

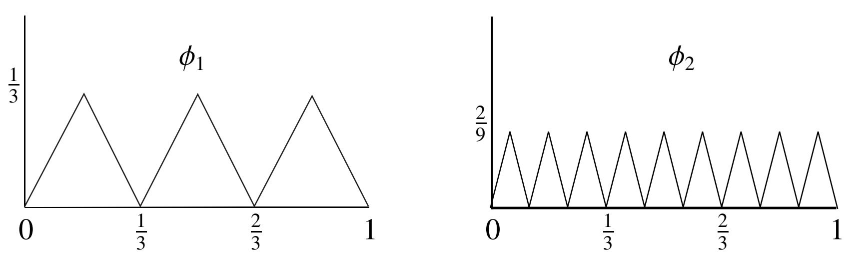

Figure 1.

Theorem 1. *Let a in $\mathbb{N}$ be greater than 2, let ϕ be the function described by Whitney, and let $t_0 = 1 - ln(2)/ln(a)$. Then ϕ has the following properties:*

1. *ϕ is not a member of Lip^α for any α satisfying $t_0 < \alpha \leq 1$; in fact, for $\tau_N = 1/(2a^N)$ the quotients*
$$\frac{|\phi(\tau_N) - \phi(0)|}{|\tau_N|^\alpha}$$
are unbounded as $N \to \infty$.
2. *ϕ belongs to Lip^β if $0 < \beta \leq t_0$; in fact, with $L = 4(a-1)/(a-2)$, it is true that $|\phi(s) - \phi(t)| \leq L|s-t|^\beta$ for all such β and for all s and t in* [0, 1].

Proof. Suppose that $t_0 < \alpha \leq 1$. If $i \leq N$, $\phi_i(\tau_N) = 2^i \phi_0(1/(2a^{N-i})/a^i = 2^i \tau_N$. If $i > N$, there are two possibilities: if a is even, $\phi_i(\tau_N) = 0$; if a is odd, $\phi_i(\tau_N) = 2^{i-1}/a^i = (2^i/a^{i-N})\tau_N$. For our purposes, it will be enough to know that $\phi_i(\tau_N) \geq 0$

for $i > N$. Then

$$\frac{|\phi(\tau_N) - \phi(0)|}{|\tau_N|^\alpha} = \frac{\sum_{i=0}^{\infty} \phi_i(\tau_N)}{\tau_N^\alpha} \geq 2^N \tau_N^{1-\alpha} \geq \frac{1}{2^{1-\alpha}} \left(\frac{2}{a^{1-\alpha}} \right)^N .$$

Since $2a^{\alpha-1} > 1$, these quotients tend to ∞ as $N \to \infty$.

Suppose next that $0 < \beta \leq t_0$. For given s and t in $[0, 1]$, let N be the nonnegative integer such that

$$\frac{1}{a^{N+1}} < |s - t| \leq \frac{1}{a^N}.$$

Note that

$$|\phi(s) - \phi(t)| \leq \sum_{i=0}^{N} |\phi_i(s) - \phi_i(t)| + \sum_{i=N+1}^{\infty} |\phi_i(s) - \phi_i(t)|$$

and that each of the functions ϕ_i is polygonal, with graph consisting of line segments of slope $\pm 2^i$ and with values in the interval $[0, (2/a)^i]$. Majorizing terms of the first sum by $2^i |s - t|$ and terms of the second by $(2/a)^i$ shows that

$$\begin{aligned}
|\phi(s) - \phi(t)| &\leq \sum_{i=0}^{N} 2^i |s - t| + (2/a)^{N+1} \sum_{i=0}^{\infty} (2/a)^i \\
&= (2^{N+1} - 1)|s - t| + (2/a)^{N+1} \frac{a}{a-2} \\
&\leq |s - t| 2^{N+1} \left(1 + \frac{a}{a-2} \right) = |s - t| 2^{N+2} \frac{a-1}{a-2}.
\end{aligned}$$

Thus,

$$\frac{|\phi(s) - \phi(t)|}{|s - t|^\beta} \leq |s - t|^{1-\beta} 2^N L \leq \left(\frac{2}{a^{1-\beta}} \right)^N L \leq L,$$

because the hypothesis on β implies that $2a^{\beta-1} \leq 1$. ■

2. THE TAKAGI–VAN DER WAERDEN FUNCTION. Since

$$\lim_{a \to \infty} 1 - \frac{\ln(2)}{\ln(a)} = 1$$

and since it *can* be shown (the proof is similar to the proof of Theorem 2) that $\lambda(\{x : \phi(x) = g(x)\}) = 0$ for each g in C^1, it would follow that one could not replace the hypothesis that f belongs to Lip^1 in Marcinkiewicz's theorem with the weaker requirement that f be in Lip^α for any specific choice of α in $(0, 1)$. However, it does not show that one could not replace that requirement with the assumption that f belongs to $\cap_{0<\gamma<1} Lip^\gamma$. It is conceivable that the leading 2^i in Whitney's description of the function ϕ_i is actually a typographical error and that Whitney was really referring to the so-called van der Waerden function

$$v(t) = \sum_{i=0}^{\infty} \frac{\phi_0(a^i t)}{a^i}$$

with parameter a in $\mathbb{N}$, $a > 1$. B. L. van der Waerden [**12**] described this function (with $a = 10$) in 1930, providing a simple example of a continuous, nowhere differentiable function. The function v was described independently by other authors, including T. Takagi [**11**] in 1903 (as was pointed out in [**1**]). We refer to it as the *Takagi–van der Waerden function*. It has been shown [**2**], [**4**] that v has more pathological nondifferentiability properties than were pointed out in [**12**]. It turns out that v does indeed satisfy the conditions described by Whitney.

Theorem 2. *The Takagi–van der Waerden function v belongs to $\bigcap_{0<\gamma<1} Lip^{\gamma}$, but v agrees with no function g from C^1 on any set of positive measure. In fact, if M is a subset of [0,1] with $\lambda(M) > 0$, then the set*

$$D(v, M) = \left\{ \frac{v(y) - v(x)}{y - x} : x \in M \ , \ y \in M \ , \ y \neq x \right\}$$

is unbounded.

Proof. The fact that v is a member of $\bigcap_{0<\gamma<1} Lip^{\gamma}$ was established by Shidfar and Sabetfakhri for $a = 2$ in [**9**], and the same fact for all integers a greater than 2 follows from the theorem in [**10**].

For each i in $\mathbb{N}$, let $f_i(t) = \phi_0(a^i t)/a^i$, and let

$$v_n(t) = \sum_{i=0}^{n} f_i(t)$$

for $0 \leq t \leq 1$. Then f_i is a polygonal function with each segment in its graph having slope ± 1, and v_n is likewise a polygonal function, each segment in its graph having slope some integer in the interval $[-n, n]$.

In order to prove the second claim in the theorem, let M be a subset of [0, 1] with $\lambda(M) > 0$ and suppose that $b > 0$ (assume without loss of generality that b is an integer). Let z in M be a Lebesgue density point of M (i.e., for every $\epsilon > 0$ there exists $\delta > 0$ such that for every subinterval $[c, d]$ of [0,1] that contains z and has length less than δ, it is true that $\lambda([c, d] \setminus M) < \epsilon \cdot (d - c)$ [**8**, pp. 12–13], [**3**, pp. 315–316]). Choose $\delta > 0$ corresponding to $\epsilon = a^{-b}/8$ in this definition, and consider an arbitrary subinterval $[c, d]$ of [0, 1] that contains z and has $|d - c| < \delta$, so that $\lambda([c, d] \setminus M) < \epsilon \cdot (d - c)$. Note that this ensures that if a subinterval (u, v) of $[c, d]$ has length $v - u \geq \epsilon \cdot (d - c)$, then (u, v) must contain a point x of M. Fix n in $\mathbb{N}$ such that $1/a^n < \delta$. We now specify $[c, d]$ to be the base (of length $1/a^n$) of one of the "roofs" of f_n that contains z. Write $m = n + b$ and $e = c + a^{-n-1}$, so that $[c, e]$ is the base of one of the "roofs" (illustrated in Figure 2) of f_{n+1}.

The polygonal function v_n has constant slope M_n over the interval $[c, e]$. There will be two cases to consider. Assume first that $M_n \geq 0$, and consider $h = c + (1/2a^m)$. Note that h is the abscissa of the top of a "roof" of f_m. The graph of v_m has constant slope $b + M_n$ on $[c, h]$. Set

$$\Delta = \frac{1}{4}(h - c) = \frac{1}{4}\frac{1}{2 \cdot a^m} = \frac{1}{8 \cdot a^{n+b}} = \frac{1}{8 \cdot a^b} \cdot \frac{1}{a^n} = \epsilon \cdot (d - c).$$

The interval (u, v) with left endpoint c and length $\Delta = \epsilon \cdot (d - c)$ must contain a point x of M. There must also exist an element y of M in the open interval of length Δ with right endpoint h (see Figure 2). We then have

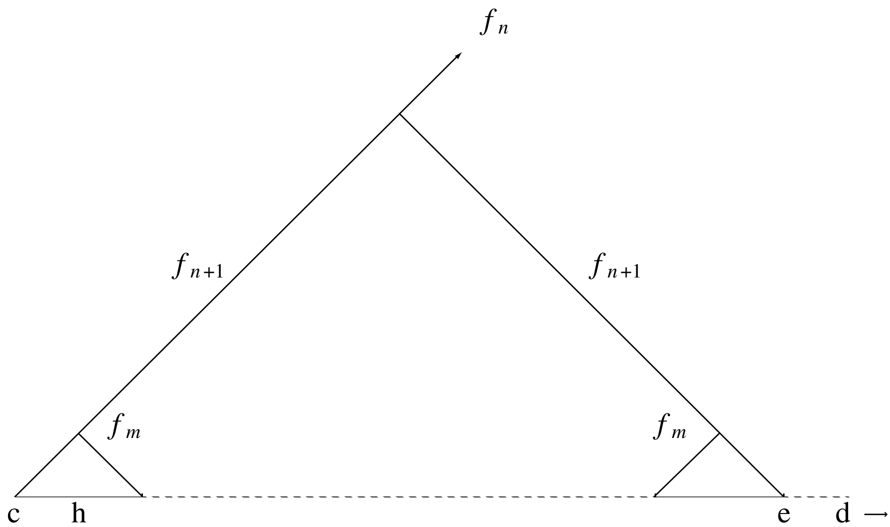

Figure 2.

$$v(y) - v(x) = v_m(y) - v_m(x) + \sum_{i=m+1}^{\infty} f_i(y) - \sum_{i=m+1}^{\infty} f_i(x),$$

so we can estimate

$$\begin{aligned} |v(y) - v(x)| &\geq (b + M_n) \cdot (y - x) - \sum_{i=m+1}^{\infty} |f_i(y) - f_i(x)| \\ &\geq (b + M_n)\frac{h - c}{2} - \sum_{i=m+1}^{\infty} \frac{1}{2 \cdot a^i} \\ &= (b + M_n)\frac{h - c}{2} - \frac{1}{2 \cdot a^{m+1}} \cdot \frac{a}{a - 1} \\ &= \left(\frac{b + M_n}{2} - \frac{1}{a - 1}\right) \cdot (h - c). \end{aligned}$$

Accordingly, for each integer $b > 0$ there exist points x and y of M for which

$$\left|\frac{v(y) - v(x)}{y - x}\right| \geq \left|\frac{v(y) - v(x)}{h - c}\right| \geq \frac{b + M_n}{2} - \frac{1}{a - 1} \geq \frac{b}{2} - \frac{1}{a - 1}.$$

Since b can be arbitrarily large, $D(v, M)$ is unbounded when $M_n \geq 0$.

The case where $M_n < 0$ is similar. Take $h = d - 1/(2a^m)$, pick an element x of M belonging to the open interval of length Δ with left endpoint h, and select a point y of M belonging the open interval of length Δ with right endpoint e. The polygonal function v_m has constant slope $-b + M_n$ on $[h, e]$. Proceeding through inequalities similar to those given in the preceding argument, one arrives at

$$\left|\frac{v(y)-v(x)}{y-x}\right| \geq \left|\frac{v(y)-v(x)}{d-h}\right| \geq \frac{b+|M_n|}{2} - \frac{1}{a-1} \geq \frac{b}{2} - \frac{1}{a-1}$$

in this case as well. Again, the unboundedness of $D(v, M)$ follows. ■

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Department of Mathematics, Auburn University, Auburn, AL 36849
brownj4@auburn.edu
kozloga@auburn.edu

When Does the Position Vector of a Space Curve Always Lie in Its Rectifying Plane?

Bang-Yen Chen

1. INTRODUCTION. Let $\mathbb{E}^3$ denote Euclidean three-space, with its inner product $\langle\ ,\ \rangle$. Consider a unit-speed space curve $\mathbf{x} : I \to \mathbb{E}^3$, where $I = (\alpha, \beta)$ is a real interval, that has at least four continuous derivatives. Let $\mathbf{t}$ denote $\mathbf{x}'$. It is possible, in general, that $\mathbf{t}'(s) = 0$ for some s; however, we assume that this never happens. Then we can introduce a unique vector field $\mathbf{n}$ and positive function κ so that $\mathbf{t}' = \kappa\mathbf{n}$. We call $\mathbf{t}'$ the *curvature vector field*, $\mathbf{n}$ the *principal normal vector field*, and κ the *curvature* of the given curve. Since $\mathbf{t}$ is a constant length vector field, $\mathbf{n}$ is orthogonal to $\mathbf{t}$. The *binormal vector field* is defined by $\mathbf{b} = \mathbf{t} \times \mathbf{n}$. It is a unit vector field orthogonal to both $\mathbf{t}$ and $\mathbf{n}$. One defines the *torsion* τ by the equation $\mathbf{b}' = -\tau\mathbf{n}$. The famous Frenet-Serret equations are given by (see, for instance, [**4**] or [**6**]):