C^1-Intersection Variant of Blumberg's Theorem

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Abstract

A C^1-Intersection Variant of Blumberg's theorem, which is related to the C-C^1 Intersection Theorem of Agronsky, Bruckner, Laczkovich, and Preiss, is proved.

1 Background

Let $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{Z}^*$ denote the positive integers, the integers, and $\mathbb{Z} \setminus \{0\}$, respectively. For $n \in \mathbb{N}$, let C, D^n, C^n, and C^∞ denote the collections of functions from $[0,1]$ into $\mathbb{R}$, that are continuous, n-times differentiable, n-times continuously differentiable, and infinitely differentiable, respectively. "D^1" (with quotation marks) denotes the class of those functions in C which are differentiable in the extended sense (i.e. $\pm\infty$ are allowed values for the derivative). If $M \subseteq [0,1]$, $C(rel\ M)$ denotes the class of continuous real valued functions with domain M (similar convention is used for the other classes). L (resp. L_0) denotes the class of Lebesgue measurable (resp. measure zero) subsets of [0,1]. λ (resp. λ^o) denotes Lebesgue measure (resp. outer measure) on [0,1]. Standard terms such as perfect sets, first category sets, sets with the Baire property, etc., whose definitions may be found in [11] will be used.

Lusin's "1st" Theorem about continuous restrictions of Lebesgue measurable functions is as follows.

Theorem 1 *For every L-measurable $f : [0,1] \to \mathbb{R}$, there exists an $M \in L \setminus L_0$ such that $f|M \in C(rel\ M)$.*

This theorem was proved by Lusin in 1912 [13] although it was already known to other researchers at the time. One can make the measure of the set M be as

[0] *Key words and phrases*: Blumberg's theorem, differentiable restriction.
[0] *1980 Mathematics Subject Classification*: Primary 26A15; Secondary 54C30.

close to 1 as is desired. However, one cannot choose M to be co-L_0 (i.e. of full measure) in [0,1]. On the other hand, Lusin showed in 1916 [14] that one could obtain a set M of full measure if one is willing to relax a bit on the niceness of the restricted function. Lusin's "2nd" Theorem about derivative restrictions of Lebesgue measurable functions is as follows.

Theorem 2 *For every L-measurable $f : [0,1] \to \mathbb{R}$, there exists an $F \in CaeD^1$ and an $M \in co\text{-}L_0$ such that $f|M = F'|M$.*

$CaeD^1$ denotes the class of continuous functions which are almost everywhere differentiable.

Blumberg's Theorem about continuous restrictions of arbitrary functions, was proved [2] in 1922.

Theorem 3 *For every $f : [0,1] \to \mathbb{R}$, there exists $M \subseteq [0,1]$, M dense in [0,1] such that $f|M \in C(rel\ M)$.*

It is clear from reading proofs of this theorem that the set M which is constructed in the proof, while dense in $[0,1]$, is nevertheless countable. It was known that the set M could not be made to have cardinality $\mathfrak{c}$ (cardinality of the continuum) because of the "Sierpinski-Zygmund function" [17] $f : [0,1] \to \mathbb{R}$ which has no continuous restriction to any set of cardinality $\mathfrak{c}$.

Consider the relationship between Lusin's "1st" and "2nd" Theorems. He was able to obtain nice restrictions of Lebesgue measurable functions to sets of full measure by relaxing the "niceness" of the restriction from continuity to the condition of being a derivative in some sense. In 1971 the current author [3] obtained a variant of Blumberg's Theorem which was of a similar flavor.

Theorem 4 *For every $f : [0,1] \to \mathbb{R}$ there exists a set $N \subseteq [0,1]$, N $\mathfrak{c}$-dense in [0,1] such that $f|N \in PWD(rel\ N)$.*

N is $\mathfrak{c}$-dense in [0,1] if every subinterval of [0,1] intersects N in a set of cardinality $\mathfrak{c}$. The class, PWD, is the collection of functions which are "pointwise discontinuous" (i.e. continuous at the elements of dense subsets of their domains). The relationship between this class, the class, B^1, of Baire-1 functions, and the class, Δ, of derivatives is indicated in the following diagram, which holds for functions with domains [0,1].

$$C \subseteq \Delta \subseteq B^1 \subseteq PWD \ .$$

Of course, it would have been preferable to have $f|N$ be in $\Delta(rel\ N)$ in the conclusion of Theorem 4 so that it would be more like Lusin's "2nd" Theorem. However, the Sierpinski-Zygmund function f [17] is constructed in such a way that f does not equal to any Borel function on any set of cardinality $\mathfrak{c}$, and the following relationship

$$\Delta(rel\ N) \subseteq B^1(rel\ N)$$

for sets N which have no isolated points was established in [15].

On the other hand, Theorem 4 has recently been improved by Katafiasz and Natkaniec [10] in that they show existence of a $\mathfrak{c}$-dense $N \subseteq [0,1]$ such that the continuity set K of $f|N$ is not only dense in N, but $f|K$ is dense in $f|N$ (as subsets of $\mathbb{R}^2$). Derivatives on [0,1] don't necessarily even have this property, which is called *super quasi continuity* (SQC) in [10].

It may have been known to researchers in the 1930's or 40's that every $f \in C$ has a restriction to a perfect set Q such that $f|Q \in D^1(rel\ Q)$ but the current author has been unable to find such a reference. The first published result of this type that the current author knows about was proved by Bruckner, Ceder, and Weiss [7] in 1969.

Theorem 5 *For every perfect $P \subseteq [0,1]$ and every $f \in C(rel\ P)$ there exists a perfect $Q \subseteq P$ such that $f|Q$ is "D^1"$(rel\ Q)$.*

Theorem 5 suggested the following variant of Blumberg's Theorem, which was proved by J. Ceder [8] in 1969.

Theorem 6 *For every uncountable $P \subseteq [0,1]$ and every function $f : P \to \mathbb{R}$ there exists $Q \subseteq P$, Q bilaterally dense in itself such that $f|Q \in$ "D^1"$(rel\ Q)$.*

A set Q is bilaterally dense (resp. $\mathfrak{c}$-dense) (resp. (non-L_0)-dense) in itself if every closed interval containing an element $x \in Q$ contains infinitely many (resp., $\mathfrak{c}$ many) (resp., a (non-L_0) subset of) elements of Q.

The following theorem about differentiable restrictions of continuous functions was proved in 1984-85.

Theorem 7 *For every perfect $P \in L \setminus L_0$ and every $f \in C(rel\ P)$, there exists a perfect $Q \subseteq P$ such that*

1. *$f|Q \in C^\infty(rel\ Q)$, and*
2. *$f|Q = g|Q$ for some $g \in C^1$.*

Theorem 7 with conclusion 1 is the C-C^∞ Restriction Theorem, proved by Laczkovich [12] in 1984, and Theorem 7 with conclusion 2, is the C-C^1 Intersection Theorem, proved by Agronsky, Bruckner, Laczkovich, and Preiss [1] in 1985. The statement of Theorem 7 would suggest the possibility that there should be analagous variants of Blumberg's Theorem. The current author has been raising this possibility as a problem in various lectures and papers [4] [5] for several years.

Problem 1 *Is it the case that for every non-L_0 $P \subseteq [0,1]$ and every $f : P \to \mathbb{R}$, there exists $Q \subseteq P$, Q bilaterally dense in itself such that*

1. $f|Q \in C^\infty(\mathit{rel}\ Q)$ *and*

2. $f|Q = g|Q$ *for some* $g \in C^1$*?*

The purpose of this paper is to present a positive solution to part 2 of the problem.

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Both parts of Theorem 7 follow as a consequence of the following theorem which was proved by Laczkovich in [12].

Theorem 8 *Let P be a perfect subset of [0,1] with $\lambda(P) > 0$ and $f : P \to \mathbb{R}$ be continuous. Then either*

1. *there exists a perfect $Q \subseteq P$ such that $f|Q$ is $D^1(\mathit{rel}\ Q)$ and $(f|Q)' \equiv 0$ or*

2. *for every $\epsilon > 0$, there exists a perfect $Q \subseteq P$ with $\lambda(P \setminus Q) < \epsilon$ such that $f|P$ is $D^1(\mathit{rel}\ Q)$.*

The following theorem is a slightly weakened Blumberg variant of the above theorem (note the difference in the measure requirement on Q given in part 2). The proof will be modelled after Laczkovich's proof of the above theorem but will necessarily be quite different because of discontinuity of the function f and nonmeasurability of the sets involved. The proof will call extensively on the Kolmogorov-Vercenko theory about *contingents* of plane sets $E \subseteq \mathbb{R}^2$. The reader is referred to Chapter IX §2 of [16] for the definitions of (1) a *half-tangent* of E at a point $a \in E$, (2) the *contingent*, $contg_E(a)$, of E at a, and (3) a *unique tangent line* of E at a. Within that context, the *length* of a subset of $\mathbb{R}^2$ is the 1-dimensional Hausdorff outer measure of the set. For notational convenience, $contg_E(a)$ will be thought of as being the collection of principal angles associated with the half-tangents of E at a rather than the collection of actual half-tangents themselves. Given a number sequence $\{x_k\}$ and a number x, $x_k \to x$ as $k \to \infty$ will be an abbreviation for $\lim_{k\to\infty} x_k = x$. $x_k \uparrow x$ as $k \to \infty$ is an abbreviation for $\{x_k\}$ strictly increasing having limit x (similar convention for $x_k \downarrow x$ as $k \to \infty$). A *box* in $\mathbb{R}^2$ is the product of two closed intervals from $\mathbb{R}$.

Theorem 9 *Let $P \subseteq [0,1]$, $\lambda^o(P) > 0$, and $f : P \to \mathbb{R}$ be arbitrary. Then either*

1. *there exists a bilaterally dense in itself $Q \subseteq P$ such that $f|Q$ is D^1(rel Q) and $(f|Q)' \equiv 0$ or*

2. *for every $\epsilon > 0$, there exists $Q \subseteq P$ with $\lambda^o(Q) > \lambda^o(P) - \epsilon$ such that $f|Q$ is D^1(rel Q).*

Proof: First, it follows from the Lebesgue density theorem it can be assumed without loss of generality that P is bilaterally (*non*-L_0)-dense in itself. Indeed, it can be assumed without loss of generality that *f is bilaterally (non-L_0)-dense in itself*, which means that for every $x \in P$ and every box $B_x = I_x \times J_x \subseteq \mathbb{R}^2$ containing $\big(x, f(x)\big)$ at the center of one of its vertical sides, the X-projection $\Pi_1(B_x \cap f)$ is *non*-L_0. To see this, let H denote the set of those points $x \in P$ for which there are intervals I_x, J_x such that $x \in I_x$, $f(x) \in \text{int}J_x$, and the projection $\Pi_1((I_x \times J_x) \cap f)$ is of measure zero. For every interval I let G_I denote the union of those open sets G for which $\Pi_1((I \times G) \cap f) \in L_0$. Then, by Lindelöf's theorem, $\Pi_1((I \times G_I) \cap f) \in L_0$. Let $\mathcal{I}$ denote the family of those closed intervals I for which $G_I \neq \emptyset$, and let $I_n \in \mathcal{I}$ $(n = 1, 2, \ldots)$ be such that $\cup_{n=1}^{\infty} I_n = \cup\mathcal{I}$. (Here we use the fact that from any system of closed intervals we can select a countable subsystem that covers each element of the system.) It is clear that

$$H \subset \bigcup_{n=1}^{\infty} \Pi_1((I_n \times G_{I_n}) \cap f),$$

and thus $H \in L_0$ (by $I_n \in \mathcal{I}$) and the elements of H can be discarded from P.

Now, let

$$A = \left\{ x \in P : \textit{both } 0 \textit{ and } \pi \ \in \ \textit{contg}_f\Big(\big(x, f(x)\big)\Big) \right\}.$$

Case I: $\lambda(P \setminus A) = 0$. Let x be any element of A. There exists $\{x_k : k \in \mathbb{Z}^*\}$ such that (1) $x_{-k} \uparrow x$, (2) $x_k \downarrow x$, (3) $\frac{f(x)-f(x_{-k})}{x-x_{-k}} \to 0$, and (4) $\frac{f(x_k)-f(x)}{x_k-x} \to 0$ as $k \to \infty$. Because we have f bilaterally (*non*-L_0)-dense in itself, we can assume without loss of generality that the x_k are chosen to lie in A. For each $k \in \mathbb{Z}^*$, let $B_k = I_k \times J_k$ be a box with center $\big(x_k, f(x_k)\big)$ such that the I_k's are disjoint and such that if $(s_k, t_k) \in B_k$ for each $k \in \mathbb{Z}^*$, then $\frac{f(x)-t_{-k}}{x-s_{-k}} \to 0$, and $\frac{t_k - f(x)}{s_k - x} \to 0$ as $k \to \infty$. This completes step 1.

Now, fix $k \in \mathbb{Z}^*$. There exists $\{x_{k,j} : j \in \mathbb{Z}^*\} \subseteq Int(I_k) \cap A$ such that (1) $x_{k,-j} \uparrow x_k$, (2) $x_{k,j} \downarrow x_k$, (3) $\frac{f(x_k)-f(x_{k,-j})}{x_k - x_{k,-j}} \to 0$, and (4) $\frac{f(x_{k,j})-f(x_k)}{x_{k,j}-x_k} \to 0$ as $j \to \infty$. Then, for each $j \in \mathbb{Z}^*$, let $B_{k,j} = I_{k,j} \times J_{k,j} \subseteq B_k$ be a box with center $\big(x_{k,j}, f(x_{k,j})\big)$ such that the $I_{k,j}$'s are disjoint and such that if $(s_{k,j}, t_{k,j}) \in B_{k,j}$ for each $j \in \mathbb{Z}^*$, then $\frac{f(x)-t_{k,-j}}{x-s_{k,-j}} \to 0$, and $\frac{t_{k,j}-f(x)}{s_{k,j}-x} \to 0$ as $j \to \infty$. This completes step 2.

Continue this process, defining for each finite sequence s of elements of $\mathbb{Z}^*$ the points x_s and boxes $B_s = I_s \times J_s$ in similar fashion. It is clear that the set

$$Q = \{x_s : s \textit{ is a finite sequence of elements of } \mathbb{Z}^*\}$$

(including $s =$ the empty sequence) satisfies the requirements of part 1 of the theorem.

Case II: $\lambda^o(P \setminus A) > 0$. Let $B = P \setminus A$ and consider the Kolmogorov-Verchenko theory applied to $f|B$. At each point a of $f|P$ either 0 or π fails to belong to $contg_{f|B}(a)$. Subdivide B into two sets B_0 and B_π, where B_0 is the set of $x \in B$ such that 0 fails to belong to $contg_{f|B}\Big((x, f(x))\Big)$ and B_π is defined similarly. Either B_0 or B_π is of positive outer measure, assume without loss of generality that it is B_0. Rotate $f|B_0$ (or the graph of $f|B_0$ if you prefer) counter clockwise $\frac{\pi}{2}$ radians about the origin in the plane and then apply the second paragraph of Lemma 3.1 on page 264 of [16] to decompose the result into countably many disjoint Lipschitz functions. Then rotate each of the pieces clockwise $\frac{\pi}{2}$ radians about the origin and the result will be to decompose B_0 into countably many disjoint parts $B_1, B_2, B_3, \ldots$ such that for each $i \in \mathbb{N}$, neither 0 nor π belongs to the contingent of any point of $f|B_i$. One of the B_i is of positive outer measure, assume without loss of generality that it is B_1. It now follows from Theorem 3.6 on page 266 of [16] that $f|B_1$ has a unique tangent line at all points except for a set of length zero. Let C be the X-projection of the set of points of $f|B_1$ with unique tangent lines and let $D \subseteq C$ be the X-projection of those points of $f|C$ where the unique tangent line is vertical. Let $E = C \setminus D$. It follows from Theorem 3.7 on page 267 of [16] that $\lambda(D) = 0$ so that $\lambda^o(E) = \lambda^o(B) > 0$. It is also clear that $f|E$ is $D^1(rel\ E)$.

It has now been shown that either 1 holds or else at least a weakend version of 2 (with $\lambda^o(Q)$ just > 0) must hold. Let $d = sup\{\lambda^o(E) : E \subseteq P \textit{ and } f|E \textit{ is } D^1(rel\ E)\}$. 2 requires that $d = \lambda^o(P)$. Assume that 1 fails. Let G be a G_δ set containing P such that $\lambda(G) = \lambda^o(P)$. Notice that if M is a measurable subset of G, $\lambda^o(M \cap P) = \lambda(M)$. For each $i \in \mathbb{N}$, let $E_i \subseteq P$ be such that $\lambda^o(E_i) > d - \frac{1}{2^i}$ and $f|E_i$ is $D^1(rel\ E_i)$ and let G_i be a G_δ subset of G such that $E_i \subseteq G_i$ and $\lambda^o(E_i) = \lambda(G_i)$. Let M_1 be a closed subset of G_1 such that $\lambda(M_1) > d - \frac{1}{2}$ and let $N_1 = M_1 \cap E_1$. Let M_2 be a closed subset of $G_2 \setminus M_1$ such that $\lambda(M_1 \cup M_2) > d - \frac{1}{2^2}$ and let $N_2 = M_2 \cap E_2$. Continuation of this process will produce a sequence $M_1, M_2, \ldots$ of disjoint closed subsets of G and a sequence $N_1, N_2, \ldots$ such that $\lambda(M_1 \cup M_2 \cup \ldots) = d$ and for each $i \in \mathbb{N}$, $N_i \subseteq M_i$, $\lambda^o(N_i) = \lambda(M_i)$, and $f|N_i$ is $D^1(rel\ N_i)$. If 2 fails (i.e. $d < \lambda^o(P)$) then $\lambda^o\big(P \setminus (M_1 \cup M_2 \cup \ldots)\big) > 0$ and it follows (still under the assumption that 1 fails) that there is a subset E of $P \setminus (M_1 \cup M_2 \cup \ldots)$ such that $\lambda^o(E) > 0$ and $f|E$ is $D^1(rel\ E)$. Choosing $n \in \mathbb{N}$ so that $\lambda(M_1 \cup M_2 \cup \ldots \cup M_n) > d - \lambda^o(E)$ it follows that if $N = N_1 \cup N_2 \cup \ldots \cup N_n \cup E$, then $N \subseteq P$, $\lambda^o(N) > d$, and $f|N$ is $D^1(rel\ N)$, which is a contradition and the theorem is proved. $\square$

Corollary 1 *Let* $P \subseteq [0,1]$*,* $\lambda^o(P) > 0$*, and* $f : P \to \mathbb{R}$ *be arbitrary. Then there exists a bilaterally dense in itself* $Q \subseteq P$ *such that* $f|Q \in C^1(rel\ Q)$*.*

Proof: Suppose $0 < \epsilon < \lambda^o(P)$. Apply Theorem 9 to f. If 1 of Theorem 9 holds, we have a conclusion stronger than the conclusion of this corollary. Otherwise, apply 2 of Theorem 9 using $\epsilon/2$ in place of ϵ to produce the set $Q = Q_1$. Now, apply Theorem 9 to f' using Q_1 in place of P. Both 1 and 2 of Theorem yield the existence of a bilaterally dense in itself $Q \subseteq P$ such that $f|Q \in D^2(rel\ Q) \subseteq C^1(rel\ Q)$ and the corollary is proved. □

In proving the C-C^1 Intersection Theorem, the authors first proved a lemma (Lemma 20 of [1]) to the effect that *if* $P \subseteq [0,1]$ *is perfect and* $f \in D^1(rel\ P)$*, then there exists a perfect* $Q \subseteq P$ *such that* $f|Q = g|Q$ *for some* $g \in C^1$. The analogous statement with 'perfect' replaced by 'bilaterally dense in itself' turns out to be false as is seen in the following.

Example 1 *There exists a bilaterally dense in itself* $P \subseteq [0,1]$ *and an* $f \in D^1(rel\ P)$ *such that if* Q *is any dense in itself subset of* P*, then* $(f|Q)'$ *is totally discontinuous and therefore* f *cannot possibly agree on* Q *with any* $g \in C^1$*.*

Proof: Let $f \in D^1$ be such that f' is a Pompeiu derivative [6], where $f' = 0$ and $f' > 0$ on two dense subsets of [0,1]. Then one can select a sequence $x_n \in [0,1]$ such that $0 < f'(x_n) < 1/n$ for every n, and $P = \{x_n : n = 1, 2, \ldots\}$ is everywhere dense in [0,1]. If $Q \subset P$ contains a convergent sequence together with its limit point x, then $f'|Q$ is not continuous at x, since $f'(x) > 0$, while the limit of $f'|P$ at x equals zero. □

On the other hand, there will still be a tool to be used in proving the next theorem.

Lemma 1 *If* $P \subseteq [0,1]$ *is bilaterally dense in itself and* $f \in C^1(rel\ P)$*, then there exists a bilaterally dense in itself* $Q \subseteq P$ *such that* $f|Q = g|Q$ *for some* $g \in C^1$*.*

Proof: Let $T = \{0,1,2\}$. Let $\delta_1 = 1$. Let $x_1 \in P$ and choose $0 < d_1 < \delta_1$ such that

$$|f'(x_1) - f'(t)| < \frac{1}{3 \cdot 2^1}$$

if $t \in P$ and $|x_1 - t| < d_1$. Now choose $0 < d_2 < d_1$ such that

$$\left|\frac{f(x_1) - f(t)}{x_1 - t} - f'(x_1)\right| < \frac{1}{3 \cdot 2^1}$$

if $t \in P$ and $|x_1 - t| < d_2$. Now choose $x_0, x_2 \in P$ satisfying $x_1 - d_2 < x_0 < x_1 < x_2 < x_1 + d_2$. It follows by combining inequalities that

$$|f'(x_i) - f'(x_j)| < \frac{2}{3 \cdot 2^1}$$

and

$$\left|\frac{f(x_i)-f(x_j)}{x_i-x_j}-f'(x_j)\right|<\frac{1}{2^1}$$

for $i \neq j \in T$. For each $i \in T$, let $B_i = I_i \times J_i \subseteq (0,1)\times(0,1)$ be a box with center $\big(x_i, f(x_i)\big)$ such that the lengths $|I_i|, |J_i|$ of the sides of the boxes are $< \frac{1}{2^1}$ and small enough that

$$\left|\frac{d-b}{c-a}-f'(e)\right|<\frac{1}{2^1}$$

if $(a,b) \in B_i$, $(c,d) \in B_j$, and $e \in I_j \cap P$ for some $i \neq j \in T$. Also make sure that the length of each of the I_j's is less than the distance between any two of the I_j's. This completes step 1 of the inductive construction process.

Suppose $n \in \mathbb{N}$ and δ_{n-1}, x_s, and $B_s = I_s \times J_s$ have been defined for each $s \in T^{n-1}$. The inductive step n is as follows. Let $\delta_n = min\{|I_s| : s \in T^{n-1}\}$. For each $s \in T^{n-1}$ rename $x_s = x_{s,1}$. Follow procedures similar to those described above to choose elements $x_{s,0} < x_{s,1} < x_{s,2}$ of $P \cap Int(I_s)$ so that

$$|f'(x_{s,i}) - f'(x_{s,j})| < \frac{2}{3\cdot 2^n} \quad \text{for} \quad i \neq j \in T \tag{1}$$

and choose boxes $B_{s,i} = I_{s,i} \times J_{s,i} \subseteq Int(B_s)$, $i \in T$, with centers $\big(x_{s,i}, f(x_{s,i})\big)$ such that the lengths $|I_{s,i}|, |J_{s,i}|$ of the sides of the boxes are $< \frac{1}{2^n}$ and small enough that

$$\left|\frac{d-b}{c-a}-f'(e)\right|<\frac{1}{2^n} \tag{2}$$

if $(a,b) \in B_{s,i}$, $(c,d) \in B_{s,j}$, and $e \in I_{s,j} \cap P$ for some $i \neq j \in T$. Also make sure that the length of each of the $I_{s,j}$'s is less than the distance between any two of the $I_{s,j}$'s. This completes step n of the inductive construction process.

Now, let $F = \bigcap_{n=1}^{\infty} \bigcup\{B_s : s \in T^n\}$, $W = \bigcap_{n=1}^{\infty} \bigcup\{I_s : s \in T^n\}$, and $Q = \bigcup_{n=1}^{\infty}\{x_s : s \in T^n\}$. Q is a bilaterally dense in itself subset of $P \cap W$, W is perfect, and F is a function (or the graph of a function if you prefer) with domain W.

For each $x \in W$ there is a unique sequence $s = \{k_n\}_{n=1}^{\infty}$ of elements of T such that x is the single element of $I_{s_1} \cap I_{s_2} \cap \ldots$, where s_n is the truncated sequence $\{k_1, k_2, \ldots, k_n\}$. It follows from inequality (1) above that the sequence $\{f'(x_{s_n})\}_{n=1}^{\infty}$ is a Cauchy sequence and its limit is denoted by $G(x)$. Notice that if $x \in Q$, $G(x)$ must equal $f'(x)$.

It will now be shown that $F \in D^1(rel\ W)$ with $F' = G$ and that the differentiability is of the "uniform" nature prescribed in simplest case of the Whitney Extension Theorem [18]. Suppose $\epsilon > 0$. Let n be such that $\frac{1}{2^n} < \frac{\epsilon}{2}$. Pick $\delta = \delta_n$ and suppose x and y belong to W and $|x-y| < \delta$. There will be a least m (necessarily $\geq n$) such x and y belong to the same I_s, where $s \in T^m$

but $x \in I_{s,i}$ and $y \in I_{s,j}$ for $i \neq j \in T$. It follows from inequality (2) above that

$$\left| \frac{F(y) - F(x)}{y - x} - f'(x_{s,i}) \right| < \frac{1}{2^m} < \frac{\epsilon}{2} \ .$$

It is also the case that

$$|f'(x_{s,i}) - G(x)| < \frac{2}{3}\left(\frac{1}{2^{m+1}} + \frac{1}{2^{m+2}} + \ \cdots \right) < \frac{2}{3}\frac{1}{2^m} < \frac{\epsilon}{2}$$

so that

$$\left| \frac{F(y) - F(x)}{y - x} - G(x) \right| < \epsilon \ .$$

In summary, it has been shown that for every $\epsilon > 0$, there exists $\delta > 0$ such that for all $x, y \in W$ for which $|y - x| < \delta$, $\left| \frac{F(y)-F(x)}{y-x} - F'(x) \right| < \epsilon$, which guarantees that F (and therefore $f|Q$) is extendable to some $g \in C^1$. This proves the Lemma. □

Corollary 1 and Lemma 1 combine to immediately yield the C^1-Intersection Variant of Blumberg's Theorem.

Corollary 2 *Let $P \subseteq [0,1]$, $\lambda^o(P) > 0$, and $f : P \to \mathbb{R}$ be arbitrary. Then there exists a bilaterally dense in itself $Q \subseteq P$ such that $f|Q = g|Q$ for some $g \in C^1$.*

Acknowledgement The author thanks the referee for many valuable suggestions. In particular, the referee simplified the first part of the proof of Theorem 9 and greatly simplified the explanation of why Example 1 has the properties described.

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