Lecture Note 2 (Ref. text book page 36)

08/18/2021

1.3 New Functions from Old Functions

In this section we start with the basic functions and obtain new functions by shifting, stretching, and reflecting their graphs. We also show how to combine pairs of functions by the standard arithmetic operations and by composition.

Transformation of Functions

Let's first consider **translations**. If c is a positive number, then the graph of $f(x)+c$ is just the graph of $y=f(x)$ shifted upward a distance of c units (because each y-coordinate is increased by the same number c). Likewise, if $g(x)=f(x-c)$, where $c>0$, then the value of g at x is the same as the value of f at $x-c$ (c units to the left of x). Therefore the graph of $y=f(x-c)$ is just the graph of $y=f(x)$ shifted c units to the right.

Vertical and Horizontal Shifts Suppose $c>0$. To obtain the graph of

$y=f(x)+c$, shift the graph of $y=f(x)$ a distance c units upward

$y=f(x)-c$, shift the graph of $y=f(x)$ a distance c units downward

$y=f(x-c)$, shift the graph of $y=f(x)$ a distance c units to the right

$y=f(x+c)$, shift the graph of $y=f(x)$ a distance c units to the left

Now let's consider the stretching and reflecting transformations. If $c>1$, then the graph of $y=cf(x)$ is the graph of $y=f(x)$ stretched by a factor of c in the vertical direction (because each y-coordinate is multiplied by the same number c). The graph of $y=-f(x)$ is the graph of $y=f(x)$ reflected about the x-axis because the point (x,y) is replaced by the point $(x,-y)$.

Vertical and Horizontal Stretching and Reflecting Suppose $c>1$. To obtain the graph of

$y=cf(x)$, stretch the graph of $y=f(x)$ vertically by a factor c

$y=(1/c)f(x)$, shrink the graph of $y=f(x)$ vertically by a factor c

$y=f(cx)$, shrink the graph of $y=f(x)$ horizontally by a factor c

$y=f(x/c)$, stretch the graph of $y=f(x)$ horizontally by a factor c

$y=-f(x)$, reflect the graph of $y=f(x)$ about the x-axis

$y=f(-x)$, reflect the graph of $y=f(x)$ about the y-axis.

Example 1 Graph the function by hand, not by plotting points, but by starting with the graph of one of the standard functions and then applying the appropriate transformations.

(a) $y=2\sqrt{x+1}$ (b) $y=2-\sqrt{x}$

Combinations of Functions

Two functions f and g can be combined to form new functions $f+g$, $f-g$, fg, and f/g in a manner similar to the way we add, subtract, multiply, and divide real numbers. The sum and difference functions are defined by

$$(f+g)(x) = f(x) + g(x) \qquad\qquad (f-g)(x) = f(x) - g(x)$$

If the domain of f is A and the domain of g is B, then the domain of $f+g$ is the intersection $A \cap B$ because both $f(x)$ and $g(x)$ have to be defined.

Similarly, the product and quotient functions are defined by

$$(fg)(x) = f(x)g(x) \qquad\qquad \left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$$

The domain of fg is $A \cap B$, but we can't divide by 0 and so the domain of f/g is $\{x \in A \cap B | g(x) \neq 0\}$.

Example 2 Find **(a)** $f+g$, $f-g$, fg, f/g and **(b)** state their domains.

$$f(x) = \sqrt{3-x}, \qquad\qquad g(x) = \sqrt{x^2-1}$$

There is another way of combining two functions to obtain a new function.

Definition Given two functions f and g, the **composite function** $f \circ g$ (also called the **composition** of f and g) is defined by

$$(f \circ g)(x) = f(g(x))$$

The domain of $f \circ g$ is the set of all x in the domain of g such that $g(x)$ is in the domain of f. In other words, $(f \circ g)(x)$ is defined whenever both $g(x)$ and $f(g(x))$ are defined.

Example 3 Find the following function and its domain

(a) $g \circ f$, where $f(x) = x^3 - 2$, $\quad g(x) = 1 - 4x$

(b) $f \circ g$, where $f(x) = \frac{x}{1+x}$, $\;g(x) = sin(2x)$