Lecture Note 4 (Ref. text book page 55) 08/23/2021

1.5 Inverse Functions and Logarithms

1 Definition A function f is called a **one-to-one function** if it never takes on the same value twice; that is,

$$f(x_1) \neq f(x_2) \qquad \text{whenever } x_1 \neq x_2$$

Thus if $f(x)$ is a one to one function, then different inputs yield different out put. The following Horizontal Line Test helps to determine if a given graph is the graph of a one-to-one function

Horizontal Line Test A function is one-to-one if and only if no horizontal line intersects its graph more than once.

Example 1. Is the function $f(x) = x^3$ one -to-one?

Example 2. Is the function $f(x) = x^2$ one -to-one?

2 Definition Let f be a one-to-one function with domain A and range B. Then its **inverse function** f^{-1} has domain B and range A and is defined by

$$f^{-1}(y) = x \iff f(x) = y$$

for any y in B.

This definition says that if f maps x into y, then f^{-1} maps y back into x. (If f were not one-to-one, then f^{-1} would not be uniquely defined.) We usually reverse the roles of x and y in Definition 2 and write

$$f^{-1}(x) = y \Leftrightarrow f(y) = x.$$

These lead to the following **cancellation equations** :

$$f^{-1}(f(x)) = x \quad \text{and} \quad f(f^{-1}(y)) = y$$

Now lets see how to compute inverse functions.

5 How to Find the Inverse Function of a One-to-One Function f

STEP 1 Write $y = f(x)$.

STEP 2 Solve this equation for x in terms of y (if possible).

STEP 3 To express f^{-1} as a function of x, interchange x and y. The resulting equation is $y = f^{-1}(x)$.

Example 3. Find the inverse function of the functions :

(a) $f(x) = 2x + 17$
(b) $f(x) = x^3 + 2.$

Logarithmic functions

The **logarithmic function with base b** and is denoted by $\log_b$ is defined to be the inverse function of the exponential function with base b. Therefore, the cancellation equations imply that

$$\boxed{\log_b(x) = y \;\Leftrightarrow\; b^y = x}$$

The logarithmic function with base e is called the **natural logarithm** and has special notation

$$\boxed{\log_e(x) = ln(x).}$$

Laws of Logarithms If x and y are positive numbers, then

1. $\log_b(xy) = \log_b x + \log_b y$

2. $\log_b\left(\frac{x}{y}\right) = \log_b x - \log_b y$

3. $\log_b(x^r) = r\log_b x$ (where r is any real number)

$$\boxed{\begin{array}{ll} \ln(e^x) = x & x \in \mathbb{R} \\ e^{\ln x} = x & x > 0 \end{array}} \qquad \boxed{9}$$

In particular, if we set $x = 1$, we get

$$\boxed{\ln e = 1}$$

Example 4 Use the laws of logarithms to evaluate $\log_2 80 - \log_2 5$.

SOLUTION Using Law 2, we have

$$\log_2 80 - \log_2 5 = \log_2\left(\frac{80}{5}\right) = \log_2 16 = 4 \qquad \text{because } 2^4 = 16.$$

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Example 5 Find x if $\ln(x) = 5$.

SOLUTION 1 From the definition we see that

$$\ln x = 5 \qquad \text{means} \qquad e^5 = x \qquad \text{Therefore } x = e^5.$$

SOLUTION 2 Start with the equation

$$\ln x = 5$$

and apply the exponential function to both sides of the equation:

$$e^{\ln x} = e^5$$

But the second cancellation equation in (9) says that $e^{\ln x} = x$. Therefore $x = e^5$.

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