Lecture Note 6 (Ref. text book page 83)

08/24/2021

2.2 The Limit of a Function

In this section we will see the intuitive meaning and definition. Let's start with an example

Example 1 Find the limit of $f(x) = \dfrac{3x^2 - 12}{x - 2}$ when $x = 2$

When we calculate the value of function at $x = 2$, it is $f(2) = \dfrac{32^2 - 12}{2 - 2} = \dfrac{0}{0}$ is undefined.
When we approach from the left side of 2 we have the following table,

x	1.5	1.8	1.9	1.99	1.999	$\cdots$	$x \to 2^-$
$f(x)$	10.5	11.4	11.7	11.97	11.997	$\cdots$	$f(x) \to 12$

When we approach from the right side of 2 we have the following table,

x	2.5	2.2	2.1	2.01	2.001	$\cdots$	$x \to 2^+$
$f(x)$	13.5	12.6	12.3	12.03	12.003	$\cdots$	$f(x) \to 12$

From the tables we see that the closer x is to 2 (on either side of 2), the closer $f(x)$ is 12. We express this by saying *the limit of the function $f(x)$ as x approaches 2 is 12.*
The notation for this is

$$\lim_{x \to 2} \frac{3x^2 - 12}{x - 2} = 12$$

Definition Limit of a Function: The function f has the limit L as x approaches a, written $\lim_{x \to a} f(x) = L$ or $f(x) \to L$ as $x \to a$ if the value of $f(x)$ can be made as close to the number L as we please by taking x sufficiently close to (but not equal to) a.

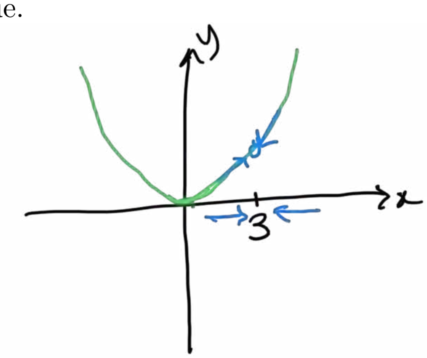
Example 2 Find the limit of the following functions at indicated value.

(a) $\lim_{x \to 3} x^2$

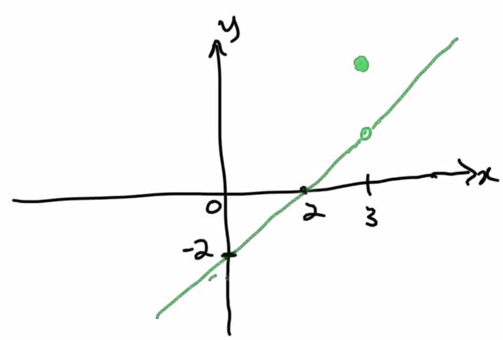
$\lim_{x \to 3^-} x^2 = 3^2 = 9$

$\lim_{x \to {}^+} x^2 = 3^2 = 9$

So $\lim_{x \to 3} x^2 = 9$



(b) $g(x) = \begin{cases} x - 2 & \text{if } x \neq 3 \\ 2 & \text{if } x = 3 \end{cases} \quad x = 3$



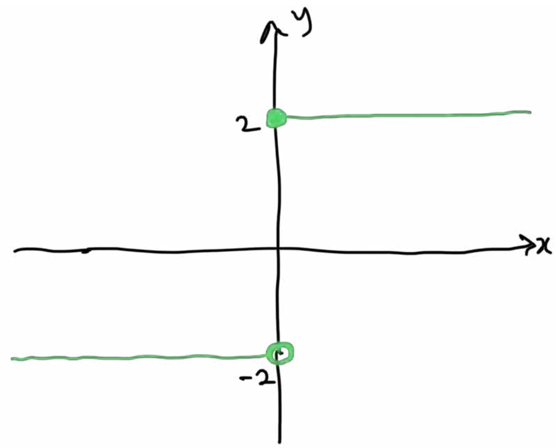
$\lim_{x \to 3^-} g(x) = 3 - 2 = 1$ and $\lim_{x \to 3^+} g(x) = 3 - 2 = 1$

So $\lim_{x \to 3} g(x) = 1.$

(c) $g(x) = \begin{cases} -2 & \text{if } x < 0 \\ 2 & \text{if } x \geq 0 \end{cases}$ $x = 0$

$\lim_{x \to 0^-} g(x) = -2$ and $\lim_{x \to 0^+} g(x) = 2$

So $\lim_{x \to 0} g(x) = \boxed{DNE}$



Theorem Let f be a function that is defined for all values of x close to $x = a$ with the possible exception of a itself. Then

$$\lim_{x \to a} f(x) = L \quad \text{if and only if} \quad \lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = L$$

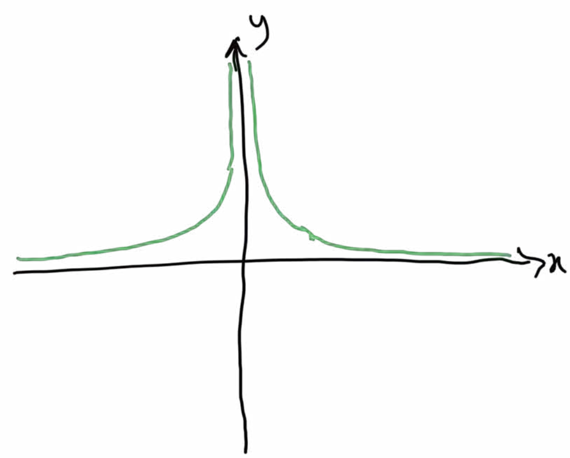
Infinite Limits

Example 3 Find $\lim_{x \to 0} \frac{1}{x^2}$ if it exists.

$\lim_{x \to 0^-} \frac{1}{x^2} = \frac{1}{0^2} = \infty$

$\lim_{x \to 0^+} \frac{1}{x^2} = \frac{1}{0^2} = \infty$

So, $\lim_{x \to 0} \frac{1}{x^2} = \infty.$



Definition The vertical line $x = a$ is called a **vertical asymptote** of the curve $y = f(x)$ if at least one of the following statements is true:

- $\lim_{x\to a} f(x) = \infty$
- $\lim_{x\to a^-} f(x) = \infty$
- $\lim_{x\to a^+} f(x) = \infty$
- $\lim_{x\to a} f(x) = -\infty$
- $\lim_{x\to a^-} f(x) = -\infty$
- $\lim_{x\to a^+} f(x) = -\infty$

Example 4. Find the vertical asymptotes of the following functions.

(a) $h(x) = \dfrac{5x}{x-5}$

$$\lim_{x\to 5^-} \frac{5x}{x-5} = \frac{5(5)}{5^- - 5} = \frac{25}{0^-} = -\infty$$

$$\lim_{x\to 5^+} \frac{5x}{x-5} = \frac{5(5)}{5^+ - 5} = \frac{25}{0^+} = \infty$$

So, $x = 5$ is a vertical asymptote of h.

(b) $f(x) = \cot x = \dfrac{\cos x}{\sin x}$, $\sin x = 0 \Rightarrow x = n\pi,\ n \in \mathbb{Z}$

So $x = n\pi$ for all $n = \dots -2, -1, 0, 1, 2, \dots$ are vertical asymptotes of f.

Since $\lim_{x\to n\pi^-} \cot x = -\infty$ and $\lim_{x\to n\pi^+} \cot x = \infty$.

(c) $g(x) = \ln(x-2)$

Here, the culprit is $x = 2$ since $\ln(x-2)\Big|_{x=2} = \ln 0$ ⚠

Also, $\text{Dom}(g) = (2, \infty)$ and

$$\lim_{x\to 2^+} \ln(x-2) = -\infty$$