Lecture Note 8 (Ref. text book page 114) 08/27/2021

2.5 Continuity

A function is continuous at a point if the graph of the function at that point is devoid of holes, gaps, jumps, or breaks.

Continuity of a Function at a Number

A function f is **continuous at a number** a if $\lim_{x \to a} f(x) = f(a)$

If f is not continuous at $x = a$, then f is said to be **discontinuous** at $x = a$. Also, f is **continuous on an interval** if f is continuous at every number in the interval.

Example 1 Find the values of x for which each function is discontinuous.

(a) $f(x) = \dfrac{x^2 - x - 2}{x - 2}$

(b) $f(x) = \begin{cases} \dfrac{1}{x^2} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$

(c) $g(x) = \begin{cases} \dfrac{x^2 - x - 2}{x - 2} & \text{if } x \neq 2 \\ 1 & \text{if } x = 2 \end{cases}$

Properties of Continuous Functions

Theorem 4: If f and g are continuous at a and c is a constant, then the following functions are also continuous at a:

1. $[f(x)]^n$, where $n \in \mathbb{R}$, is continuous at $x = a$ whenever it is defined at that number.
2. $f \pm g$
3. cf
4. fg
5. f/g, provided that $g(a) \neq 0$.

Note: 1. The constant function $f(x) = c$ is continuous every where.

2. The identity function $f(x) = x$ is continuous every where.

Theorem 7: The following types of functions are continuous at every number in their domains: polynomials, rational functions, root functions, trigonometric functions, inverse trigonometric functions, exponential functions, & logarithmic functions.

Example 2 Find the values of x for which each function is continuous.

(a) $f(x) = 7x^4 - 3x^2 + 5x - 8$

(b) $g(x) = \dfrac{5x^{11} + 3x^2 - 6}{x^2 + 4}$

(c) $h(x) = \dfrac{x^3 + 2x^2 - 1}{5 - 3x}$

(d) $k(x) = \dfrac{\ln(x) + \tan^1 x}{x^2 - 1}$

Example 3 Evaluate $\displaystyle\lim_{x \to \pi} \frac{\sin x}{2 + \cos x}$.

The Intermediate Value Theorem

If f is a continuous function on a closed interval $[a, b]$ and M is any number between $f(a)$ and $f(b)$, then there is at least one number c in $[a, b]$ such that $f(c) = M$.

Existence of Zeros of a Continuous Function

If f is a continuous function on a closed interval $[a, b]$, and if $f(a)$ and $f(b)$ have opposite signs, then there is at least one solution of the equation $f(x) = 0$ in the interval (a, b).

Example 4 Let $f(x) = 4x^3 - 6x^2 + 3x - 2$

(a) Show that f is continuous for all values of x.

(b) Show that there exists a number $x = c$ where $c \in (1,\ 2)$ and $f(c) = 0$.