Lecture Note 9 (Ref. text book page 126) 08/30/2021

2.6 Limits at Infinity; Horizontal Asymptotes

In this section we investigate the limit of a function at infinity.

The Limit of a Function at Infinity

The function f has the limit L as x increases without bound (or as x approaches infinity), written

$$\lim_{x\to\infty} f(x) = L$$

if $f(x)$ can be made arbitrarily close to L by taking x large enough.

Similarly, the function f has the limit M as x decreases without bound (or as x approaches negative infinity), written

$$\lim_{x\to-\infty} f(x) = M$$

if $f(x)$ can be made arbitrarily close to M by taking x to be negative and sufficiently large in absolute value.

Note

(a) $\infty + \infty = \infty$

(b) $\infty \cdot \infty = \infty$

(c) $\infty^n = \infty$, $n > 0$

(d) $c \cdot \infty = \infty$, $c > 0$

(e) $0/0$ is undefined

(f) $0/c = 0$, $c \neq 0$

(g) $c/0$ is undefined

(h) $c/\infty = 0$

(i) $\infty - \infty$ is undefined

(j) ∞/∞ is undefined

(k) 1^∞ is undefined

Definition The line $y = L$ is called a **horizontal asymptote** of the curve $y = f(x)$ if either

$$\lim_{x\to\infty} f(x) = L \quad \text{or} \quad \lim_{x\to-\infty} f(x) = L$$

Example 1 Sketch the graph of an example of a function f that satisfies all of the given conditions.

- $\lim_{x\to 2} f(x) = \infty$
- $\lim_{x\to -2^+} f(x) = \infty$
- $\lim_{x\to -2^-} f(x) = -\infty$
- $\lim_{x\to -\infty} f(x) = 0$
- $\lim_{x\to \infty} f(x) = 0$
- $f(0) = 0$

Theorem If $r > 0$ is a rational number then $\lim_{x\to\infty} \frac{1}{x^r} = 0$, $\lim_{x\to-\infty} \frac{1}{x^r} = 0$ for $r > 0$ such that x^r is defined.

Example 2 Find the limit or show that it does not exist.

(a) $\lim_{x\to\infty} \frac{\sqrt{9x^6 - x}}{x^3 + 1}$ (b) $\lim_{x\to\infty} \frac{3x^2 - x - 2}{5x^2 + 4x + 1}$ (c) $\lim_{x\to-\infty} \frac{x - 2}{x^2 + 1}$

Example 3 Compute

(a) $\lim_{x\to\infty} (\sqrt{x^2 + 1} - x)$ (b) $\lim_{x\to\infty} (x^2 - x)$

Example 4 Find the limit or show that it does not exist.

(a) $\lim_{x\to\infty} \frac{1 - e^x}{1 + 2e^x}$ (b) $\lim_{x\to\infty} (e^{-2x} cos x)$ (c) $\lim_{x\to\infty} [\ln(1 + x^2) - \ln(1 + x)]$