Lecture Note 11 (Ref. text book page 152) 09/03/2021

2.8 The Derivative as a Function

In the preceding section we considered the derivative of a function f at a fixed number a:

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \qquad (\star)$$

Here we change our point of view and let the number a vary. If we replace a in Equation $(\star)$ by a variable x, we obtain the derivative of f as

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Example 1 Find the derivative of the following functions. State the domain of the function and the domain of its derivative.

(a) $f(x) = x^3 - x$ (b) $f(x) = \sqrt{9-x}$ (c) $f(x) = \dfrac{1-x}{2+x}$

Note: If we use the traditional notation $y = f(x)$ to indicate that the independent variable is x and the dependent variable is y, then some common alternative notations for the derivative are as follows:

$$f'(x) = y' = \frac{dy}{dx} = \frac{d}{dx}f(x) = Df(x) = D_x f(x)$$

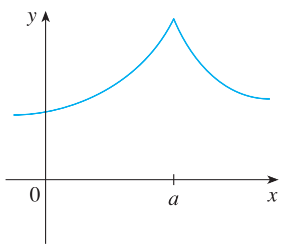
A function f is **differentiable at** a if $f'(a)$ exists. It is **differentiable on an open interval** (a, b) [or (a, ∞) or $(-\infty, a)$ or $(-\infty, \infty)$] if it is differentiable at every number in the interval.

Example 2 Where is the function $f(x) = |x|$ differentiable?

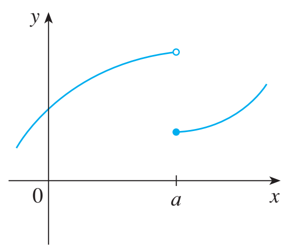
Theorem If f is differentiable at a, then f is continuous at a.

The converse of the theorem is false!

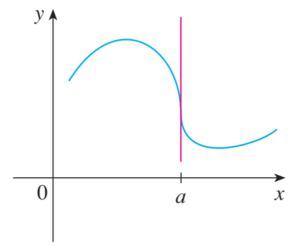
How can a Function Fail To Be Differentiable? Three ways for f not to be differentiable at a: (i) If the graph of function f has a "corner" or "kink" in it, then the graph of f has no tangent at this point and f is not differentiable there. (ii) From the theorem we have that if f is not continuous at a, then f is not differentiable at a. (iii) A third possibility is that the curve has a **vertical tangent line** when $x = a$; that is, f is continuous at a and $\lim_{x\to a} |f'(x)| = \infty$.



(a) A corner



(b) A discontinuity



(c) A vertical tangent

If f is a differentiable function, then its derivative f' is also a function, so f' may have a derivative of its own, denoted by $(f')' = f''$. This new function f'' is called the **second derivative** of f because it is the derivative of the derivative of f.

Example 3 Find the second and third derivative of $f(x) = x^3 - x$.