Lecture Note 20 (Ref. text book page 251) 10/05/2021

3.10 Linear Approximations and Differentials

Linearization

The idea is that it might be easy to calculate a value $f(a)$ of a function, but difficult (or even impossible) to compute nearby values of f. So we settle for the easily computed values of the linear function L whose graph is the tangent line of f at $(a, f(a))$.

The following function is called the **linear approximation** or **tangent line approximation** of f at a,

$$f(x) \approx f(a) + f^{'}(a)(x-a)$$

The linear functions whose graph is this tangent line, that is,

$$L(x) = f(a) + f^{'}(a)(x-a)$$

is called the **linearization** of f at a.

Example 1 Find the linearization $L(x)$ of the function $f(x) = \sin x$ at $\pi/6$.

Differentials

The ideas behind linear approximations are sometimes formulated in the terminology and notation of differentials. if $y = f(x)$, where f is a differentiable function, then the **differential** dx is an independent variable; that is, dx can be given the value of any real number.

The **differential** dy is then defined in terms of dx by the equation

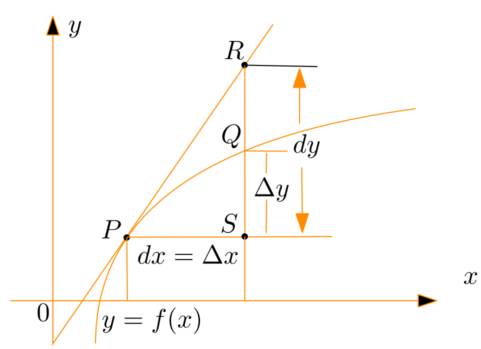
$$dy = f^{'}(x)dx$$

So dy is a dependent variable; it depends on the values of x and dx.

The geometric meaning of differentials is shown in the Figure. Let $P(x, f(x))$ and $Q(x+\Delta x, f(x+\Delta x))$ be points on the graph of f and let $dx = \Delta x$. The corresponding change in y is

$$\Delta y = f(x+\Delta x) - f(x)$$

The slope of the tangent line PR is the derivative $f^{'}(x)$. Thus the directed distance from S to R is $f^{'}(x)dx = dy$. Therefore dy represents the amount that the tangent line rises or falls (the change in the linearization), whereas Δy represents the amount that the curve $y = f(x)$ rises or falls when x changes by an amount dx.



Example 2 Find the differential of each function.

(a) $y = \dfrac{1+2u}{1+3u}$

(b) $y = \theta^2 \sin 2\theta$

Exampla 3 Use a linear approximation (or differentials) to estimate the given number.

$$(1.999)^5$$

Exampla 4 Use a linear approximation (or differentials) to estimate the given number.

$$e^{-0.01}$$