Lecture Note 13 (Ref. text book page 190) 09/15/2021

3.3 Derivatives of Trigonometric Functions

In this section we learn how to differentiate trigonometric functions.

(i) $\frac{d}{dx}(\sin x) = \cos x$ (iv) $\frac{d}{dx}(\csc x) = -\csc x \cot x$

(ii) $\frac{d}{dx}(\cos x) = -\sin x$ (v) $\frac{d}{dx}(\sec x) = \sec x \tan x$

(iii) $\frac{d}{dx}(\tan x) = \sec^2 x$ (vi) $\frac{d}{dx}(\cot x) = -\csc^2 x$

Example 1 Find the derivative of the following functions by using the rules of differentiation.

(a) $f(x) = x\cos x + 2\tan x$

$f'(x) = \frac{d}{dx}(x\cos x) + 2\frac{d}{dx}(\tan x)$

$= \boxed{\cos x - x\sin x + 2\sec^2 x}$

(c) $f(t) = te^t \cot t$

$u = te^t \Rightarrow u' = \frac{d}{dt}(te^t)$

$= e^t + te^t$

$v = \cot t \Rightarrow v' = -\csc^2 t$

So

$f'(t) = u'v + uv'$

$= (e^t + te^t)\cot t - te^t\csc^2 t$

$= \boxed{e^t\left[(1+t)\cot t - t\csc^2 t\right]}$

(b) $g(\theta) = e^\theta(\tan\theta - \theta)$

$u = e^\theta \Rightarrow u' = e^\theta$

$v = \tan\theta - \theta \Rightarrow v' = \sec^2\theta - 1$

So,

$g'(\theta) = u'v + uv'$

$= e^\theta(\tan\theta - \theta) + e^\theta(\sec^2\theta - 1)$

$= \boxed{e^\theta\left[\tan\theta - \theta + \sec^2\theta - 1\right]}$

(d) $y = \frac{\cos x}{1 - \sin x}$

$u = \cos x \Rightarrow u' = -\sin x$

$v = 1 - \sin x \Rightarrow v' = -\cos x$

So,

$y' = \frac{u'v - uv'}{v^2}$

$= \frac{-\sin x(1-\sin x) - \cos x(-\cos x)}{(1-\sin x)^2}$

$= \frac{-\sin x + \sin^2 x + \cos^2 x}{(1-\sin x)^2}$

$= \boxed{\frac{-\sin x}{(1-\sin x)^2}}$

Example 2 For what values of x does the graph of $f(x) = e^x \cos x$ have a horizontal tangent?

$f'(x) = e^x \cos x - e^x \sin x = 0$

$\Rightarrow e^x(\cos x - \sin x) = 0$

$\Rightarrow \cos x - \sin x = 0$ since $e^x \neq 0$

$\Rightarrow \cos x = \sin x$

$\dfrac{\cos x}{\cos x} = \dfrac{\sin x}{\cos x}$

$\Rightarrow 1 = \tan x$

$\Rightarrow \boxed{x = \dfrac{\pi}{4} + n\pi}$

Example 3 Find an equation of the tangent line to the curve $y = e^x \cos x$ at the point $(0, 1)$

$y - y_1 = m(x - x_1)$

$y - 1 = 1(x - 0)$

$\boxed{y = x + 1}$

$m = y'(x)\Big|_{x=0}$

$= (e^x \cos x)'\Big|_{x=0}$

$= e^x \cos x - e^x \sin x\Big|_{x=0}$

$= e^0 \cos 0 - e^0 \sin 0$

$= 1$

Example 4 If $f(t) = \sec t$, find $f''(\pi/4)$

$f(t) = \sec t = \dfrac{1}{\cos t}$

$\Rightarrow f'(t) = \dfrac{\cos t(0) - 1\cdot(-\sin t)}{\cos^2 t}$

$= \dfrac{\sin t}{\cos^2 t} = \tan t \sec t$

$f''(t) = u'v + uv'$

$= \sec^2 t \sec t - \tan t(\sec t \tan t)$

$= \sec^3 t - \tan^2 t \sec t$

$f''(\pi/4) = (\sec \pi/4)^3 - (\tan \pi/4)^2 \sec \pi/4$

$= (\sqrt{2})^3 - (1)^2(\sqrt{2})$

$= 2\sqrt{2} - \sqrt{2} = \boxed{\sqrt{2}}$

Table of Differentiation Formulas

(1.) $\frac{d}{dx}(c) = 0$ **(2.)** $\frac{d}{dx}(x^n) = nx^{n-1}$ **(3.)** $\frac{d}{dx}(e^x) = e^x$

(4.) $\frac{d}{dx}(cf) = cf'$ **(5.)** $(f+g)' = f' + g'$ **(6.)** $(f-g)' = f' - g'$

(7.) $\frac{d}{dx}(fg) = fg' + gf'$ **(8.)** $\left(\frac{f}{g}\right)' = \frac{gf' - fg'}{g^2}$ **(9.)** $\left(e^{f(x)}\right)' = f'(x)e^{f(x)}$

(10) $\frac{d}{dx}(g(x))^n = n(g(x))^{n-1} \cdot g'(x)$ **(11.)** $\frac{d}{dx}\big(tan(x)\big) = \sec^2(x)$ **(12.)** $\frac{d}{dx}\big(cot(x)\big) = -\csc^2(x)$

(13.) $\frac{d}{dx}\cos(f(x)) = -f'(x)\sin(x)$ **(14.)** $\frac{d}{dx}\sin(u(x)) = f'(x)\cos(f(x))$

Let's try to confirm our guess that if $f(x) = \sin x$, then $f'(x) = \cos x$. From the definition of a derivative, we have

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h}$$

$$= \lim_{h \to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h}$$

$$= \lim_{h \to 0} \left[\frac{\sin x \cos h - \sin x}{h} + \frac{\cos x \sin h}{h} \right]$$

$$= \lim_{h \to 0} \left[\sin x \left(\frac{\cos h - 1}{h} \right) + \cos x \left(\frac{\sin h}{h} \right) \right]$$

1 $$= \lim_{h \to 0} \sin x \cdot \lim_{h \to 0} \frac{\cos h - 1}{h} + \lim_{h \to 0} \cos x \cdot \lim_{h \to 0} \frac{\sin h}{h}$$

Two of these four limits are easy to evaluate. Since we regard x as a constant when computing a limit as $h \to 0$, we have

$$\lim_{h \to 0} \sin x = \sin x \qquad \text{and} \qquad \lim_{h \to 0} \cos x = \cos x$$

2 $$\boxed{\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1}$$

We can deduce the value of the remaining limit in (1) as follows:

$$\lim_{\theta \to 0} \frac{\cos \theta - 1}{\theta} = \lim_{\theta \to 0} \left(\frac{\cos \theta - 1}{\theta} \cdot \frac{\cos \theta + 1}{\cos \theta + 1} \right) = \lim_{\theta \to 0} \frac{\cos^2\theta - 1}{\theta (\cos \theta + 1)}$$

$$= \lim_{\theta \to 0} \frac{-\sin^2\theta}{\theta (\cos \theta + 1)} = -\lim_{\theta \to 0} \left(\frac{\sin \theta}{\theta} \cdot \frac{\sin \theta}{\cos \theta + 1} \right)$$

$$= -\lim_{\theta \to 0} \frac{\sin \theta}{\theta} \cdot \lim_{\theta \to 0} \frac{\sin \theta}{\cos \theta + 1}$$

$$= -1 \cdot \left(\frac{0}{1 + 1} \right) = 0 \qquad \text{(by Equation 2)}$$

3 $$\boxed{\lim_{\theta \to 0} \frac{\cos \theta - 1}{\theta} = 0}$$

If we now put the limits (2) and (3) in (1), we get

$$f'(x) = \lim_{h \to 0} \sin x \cdot \lim_{h \to 0} \frac{\cos h - 1}{h} + \lim_{h \to 0} \cos x \cdot \lim_{h \to 0} \frac{\sin h}{h}$$

$$= (\sin x) \cdot 0 + (\cos x) \cdot 1 = \cos x$$

So we have proved the formula for the derivative of the sine function:

4 $$\boxed{\frac{d}{dx} (\sin x) = \cos x}$$