Section 3.4

(7) $F(x) = (5x^6+2x^3)^4 = (u(x))^4$ where $u(x) = 5x^6+2x^3$

$\Rightarrow u'(x) = 30x^5+6x^2$

So

$$\begin{aligned}
F'(x) &= F'(u)\,u'(x) \\
&= (4u^3)(30x^5+6x^2) \\
&= 4(5x^6+2x^3)^3(30x^5+6x^2) \\
&= 4(x^3(5x^3+2))^3(6x^2(5x^3+1)) \\
&= 4\cdot x^9\cdot 6x^2(5x^3+2)^3(5x^3+1) \\
&= \underline{\underline{24x^{11}(5x^3+2)^3(5x^3+1)}}
\end{aligned}$$

(16) $g(x) = e^{x^2-x} = e^u$ where $u(x) = x^2-x$

$\Rightarrow u'(x) = 2x-1$

So

$$\begin{aligned}
g'(x) &= g'(u)\cdot u'(x) \\
&= e^u(2x-1) \\
&= \underline{\underline{e^{x^2-x}(2x-1)}}
\end{aligned}$$

(17) $F(x) = (2x-3)^4(x^2+x+1)^5$

(17) $f(x) = (2x-3)^4 (x^2+x+1)^5$

$= u(x)\, v(x)$ where $u(x) = (2x-3)^4$ and $v(x) = (x^2+x+1)^5$

Now,

$u(x) = (2x-3)^4 = (w)^4$ where $w(x) = 2x-3 \Rightarrow w'(x) = 2$

$\Rightarrow u'(x) = u'(w)\cdot w'(x) = 4w^3 \cdot 2 = 8(2x-3)^3$

and

$v(x) = (x^2+x+1)^5 = (w)^5$ where $w(x) = x^2+x+1 \Rightarrow w'(x) = 2x+1$

$\Rightarrow v'(x) = v'(w)\cdot w'(x) = 5w^4 \cdot (2x+1) = 5(x^2+x+1)^4 (2x+1)$

Thus, by product rule:

$f'(x) = u'(x)v(x) + u(x)v'(x)$

$= 8(2x-3)^3(x^2+x+1)^5 + (2x-3)^4\left(5(x^2+x+1)^4(2x+1)\right)$

$= (2x-3)^3(x^2+x+1)^4\left(8x^2+8x+8 + 5(2x-3)(2x+1)\right)$

$= (2x-3)^3(x^2+x+1)^4\left(8x^2+8x+8 + 20x^2+10x-30x-15\right)$

$= (2x-3)^3(x^2+x+1)^4\left(28x^2-12x-7\right)$

(25) $g(u) = \left(\frac{u^3-1}{u^3+1}\right)^8 = (w)^8$ where $w(u) = \frac{u^3-1}{u^3+1}$

$\Rightarrow w'(u) = \frac{(u^3+1)(3u^2) - (u^3-1)(3u^2)}{(u^3+1)^2} = \frac{3u^5+3u^2-3u^5+3u^2}{(u^3+1)^2} = \frac{6u^2}{(u^3+1)^2}$

Thus,

Thus,

$$g'(u) = g'(w) \cdot w'(u)$$

$$= 8w^7 \cdot \left(\frac{6u^2}{(u^3+1)^2} \right)$$

$$= 8\left(\frac{u^3-1}{u^3+1}\right)^7 \left(\frac{6u^2}{(u^3+1)^2}\right)$$

$$= \frac{48u^2 (u^3-1)^7}{(u^3+1)^7 (u^3+1)^2}$$

$$= \frac{48u^2(u^3-1)^7}{(u^3+1)^9}$$

//

31 $F(t) = e^{t \sin 2t} = e^u$ where $u(t) = t \sin 2t$

$$\Rightarrow u'(t) = t(2\cos 2t) + 1 \cdot \sin 2t$$

$$= 2t\cos 2t + \sin 2t$$

So

$$F'(t) = F'(u) \cdot u'(t)$$

$$= e^u \cdot (2t\cos 2t + \sin 2t)$$

$$= e^{t\sin 2t}(2t\cos 2t + \sin 2t)$$

37 $y = \cot^2(\sin\theta)$

$$= (\cot(\sin\theta))^2$$

$= (\cot(\sin\theta))^2$

$= u^2$ where $u(\theta) = \cot(\sin\theta)$

$= \cot w$ where $w(\theta) = \sin\theta$

$\Rightarrow w'(\theta) = \cos\theta$

$\Rightarrow u'(\theta) = u'(w)\cdot w'(\theta)$

$= -\csc^2 w \cdot \cos\theta$

$= -\csc^2(\sin\theta)\cdot \cos\theta$

Thus,

$y'(\theta) = y'(u)\cdot u'(\theta)$

$= 2u\cdot\left(-\csc^2(\sin\theta)\cdot\cos\theta\right)$

$= -2\cos\theta\,(\cot(\sin\theta))(\csc^2(\sin\theta))$

45 $y = \cos\sqrt{\sin(\tan\pi x)}$

$= \cos u \Rightarrow y'(x) = y'(u)\cdot u'(x) = -\sin u\cdot u'(x)$

where

$u(x) = \sqrt{\sin(\tan\pi x)}$

$= v^{1/2} \Rightarrow u'(x) = u'(v)\cdot v'(x) = \frac{1}{2}v^{-1/2}\cdot v'(x)$

where $v(x) = \sin(\tan\pi x)$

$= \sin w \Rightarrow v'(x) = \cos w \cdot w'(x) = \pi\sec^2\pi x\cos(\tan\pi x)$

where $w(x) = \tan\pi x \Rightarrow w'(x) = \pi\sec^2\pi x$

Hence,

$$y' = -\sin u \cdot u'(x)$$

$$= -\sin\left(\sqrt{\sin(\tan\pi x)}\right) \cdot \tfrac{1}{2} v^{-1/2} \cdot v'(x)$$

$$= \frac{-\sin\left(\sqrt{\sin(\tan\pi x)}\right) v'(x)}{2\sqrt{v}}$$

$$= \frac{-\sin\left(\sqrt{\sin(\tan\pi x)}\right)\left(\pi \sec^2\pi x \cos(\tan\pi x)\right)}{2\sin(\tan\pi x)}$$

$$= \frac{-\pi \sin\left(\sqrt{\sin(\tan\pi x)}\right)\left(\sec^2\pi x \cos(\tan\pi x)\right)}{2\sin(\tan\pi x)}$$

47 $y = \cos(\sin 3\theta)$

$= \cos u$ where $u(\theta) = \sin 3\theta \Rightarrow u'(\theta) = 3\cos 3\theta$

So

$$y'(\theta) = y'(u) \cdot u'(\theta)$$

$$= -\sin u \cdot (3\cos 3\theta)$$

$$= -3\cos 3\theta \sin(\sin 3\theta)$$

$$y'(\theta) = -3\cos 3\theta \sin(\sin 3\theta)$$
$$= u(\theta)\, v(\theta)$$

where $u(\theta) = -3\cos 3\theta \Rightarrow u'(\theta) = 9\sin 3\theta$ and $v(\theta) = \sin(\sin 3\theta)$
$$= \sin w$$

where $w(\theta) = \sin 3\theta \Rightarrow w'(\theta) = 3\cos 3\theta$, so

$$v'(\theta) = v'(w)\cdot w'(\theta)$$
$$= \cos w \cdot 3\cos 3\theta$$
$$= 3\cos 3\theta \cos(\sin 3\theta)$$

Hence,

$$y''(\theta) = u'(\theta)v(\theta) + u(\theta)v'(\theta)$$
$$= 9\sin 3\theta\left(\sin(\sin 3\theta)\right) + (-3\cos 3\theta)\left(3\cos 3\theta \cos(\sin 3\theta)\right)$$
$$= 9\sin 3\theta\left(\sin(\sin 3\theta)\right) - 9\cos^2 3\theta \cos(\sin 3\theta)$$

71 Let $r(x) = f(g(h(x)))$, where $h(1) = 2$, $g(2) = 3$, $h'(1) = 4$, $g'(2) = 5$, and $f'(3) = 6$. Find $r'(1)$.

$$r'(x) = f'(g)\cdot g'(h)\cdot h'(x)$$
$$= f'(g(h(x)))\cdot g'(h(x))\cdot h'(x)$$

So
$$r'(1) = f'(g(h(1)))\cdot g'(h(1))\cdot h'(1)$$
$$= f'(g(2))\cdot g'(2)\cdot 4$$

$$= f'(3) \cdot 5 \cdot 4$$

$$= 6 \cdot 5 \cdot 4$$

$$= 120$$

77) $y = \cos 2x$, $y^{(50)} = ?$

$$y' = -2\sin 2x = -2^1 \sin 2x$$

$$y'' = -4\cos 2x = -2^2 \cos 2x$$

$$y''' = 8\sin 2x = 2^3 \sin 2x$$

$$y^{iv} = 16 \cos 2x$$

$$= 2^4 \cos 2x$$

So $\cos 2x$ resurfaces after every 4th derivative!

But

$$50 = 12(4) + 2$$

$\Rightarrow$ y'' and $y^{(50)}$ have the same form except for the positive constant multiple

$$\Rightarrow y^{(50)} = -2^{50} \cos 2x$$