Lecture Note 16 (Ref. text book page 218)

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3.6 Derivatives of Logarithmic Functions

In this section we use implicit differentiation to find the derivatives of the logarithmic functions $y = \log_b x$ and, in particular, the natural logarithmic function $y = \ln x$.

$$\frac{d}{dx}(\log_b x) = \frac{1}{x \ln b} \quad \Rightarrow \quad \frac{d}{dx}(\ln x) = \frac{1}{x}$$

So by the chain rule we can generate the formula as follow

$$\frac{d}{dx}(\ln u) = \frac{u'}{u} \quad \text{or} \quad \frac{d}{dx}[\ln g(x)] = \frac{g'(x)}{g(x)}$$

where u is a function of x.

Example 1 Find the derivative of the following functions.

(a) $f(x) = \sin(\ln x)$ (b) $y = \ln(e^{-x} + xe^{-x})$ (c) $g(x) = \log_2(x \log_5 x)$

Logarithmic Differentiation

The calculation of derivatives of complicated functions involving products, quotients, or powers can often be simplified by taking logarithms. The steps of logarithmic differentiation is as following;

1. Take natural logarithms of both sides of an equation $y = f(x)$ and use the Laws of Logarithms to simplify.
2. Differentiate implicity with respect to x.
3. Solve the resulting equation for y'.

Example 2 Use logarithmic differentiation to find the derivative of the function.

(a) $y = (x^2 + 2)^2(x^4 + 4)^4$

(b) $y = (\ln x)^{\cos x}$

Example 3 Find y' if $x^y = y^x$.

Example 1 Find the derivative of the following functions.

(a) $f(x) = \sin(\ln x)$

$= \sin u,\ u = \ln x$

$\Rightarrow f'(x) = f'(u)\cdot u'(x)$

$= \cos u \cdot \frac{1}{x}$

$= \boxed{\dfrac{\cos(\ln x)}{x}}$

(b) $y = \ln(e^{-x} + xe^{-x})$

$= \ln u,\ u = e^{-x} + xe^{-x}$

$\Rightarrow y' = y'(u)\cdot u'(x)$

$= \frac{1}{u}\cdot u'(x)$

$= \dfrac{u'(x)}{u}$

$= \dfrac{-e^{-x} + e^{-x} - xe^{-x}}{e^{-x} + xe^{-x}}$

$= \dfrac{-xe^{-x}}{e^{-x} + xe^{-x}}$

(c) $g(x) = \log_2(x \log_5 x)$

$y = \log_2 u,\ u = x\log_5 x$

$\Rightarrow 2^y = u$

$\Rightarrow \ln 2^y = \ln u$

$\Rightarrow y\ln 2 = \ln u \Rightarrow y = \dfrac{\ln u}{\ln 2}$

$\Rightarrow y'(x) = \dfrac{u'(x)}{u\ln 2}$

But by product rule,

$u'(x) = \log_5 x + \frac{1}{\ln 5}$

So,

$g'(x) = \boxed{\dfrac{\log_5 x + \frac{1}{\ln 5}}{x(\ln 5)(\log_5 x)}}$

Example 2 Use logarithmic differentiation to find the derivative of the function.

(a) $y = (x^2+2)^2(x^4+4)^4$

$\Rightarrow \ln y = \ln (x^2+2)^2(x^4+4)^4$

$= \ln(x^2+2)^2 + \ln(x^4+4)^4$

$= 2\ln(x^2+2) + 4\ln(x^4+4)$

$\Rightarrow \frac{d}{dx}(\ln y) = \frac{d}{dx}\left[2\ln(x^2+2) + 4\ln(x^4+4)\right]$

$\Rightarrow \dfrac{y'}{y} = 2\cdot\dfrac{2x}{x^2+2} + 4\cdot\dfrac{4x^3}{x^4+4}$

$\Rightarrow y' = y\left[\dfrac{4x}{x^2+2} + \dfrac{16x^3}{x^4+4}\right]$

Replace y

(b) $y = (\ln x)^{\cos x}$

$\Rightarrow \ln y = \ln(\ln x)^{\cos x} = \cos x\,(\ln(\ln x))$

i.e. $\ln y = \cos x\,(\ln(\ln x))$

Using product rule on the RHS,

$\dfrac{y'}{y} = -\sin x\,(\ln(\ln x)) + \cos x\left(\dfrac{(\ln x)'}{\ln x}\right)$

$y' = y\left[-\sin x\,(\ln(\ln x)) + \dfrac{\cos x}{x\ln x}\right]$

$= \boxed{(\ln x)^{\cos x}\left[\dfrac{\cos x}{x\ln x} - \sin x\,(\ln(\ln x))\right]}$

Example 3 Find y' if $x^y = y^x$.

$\Rightarrow \ln x^y = \ln y^x \Rightarrow y\ln x = x\ln y.$

Using product rule and implicit differentiation,

$y\left(\frac{1}{x}\right) + y'\ln x = x\left(\frac{y'}{y}\right) + \ln y \Rightarrow y^2 + xyy'\ln x = x^2y' + xy\ln y$

$\Rightarrow xyy'\ln x - x^2y' = xy\ln y - y^2 \Rightarrow (xy\ln x - x^2)y' = xy\ln y - y^2$

Hence,

$y' = \boxed{\dfrac{xy\ln y - y^2}{xy\ln x - x^2}}$