Lecture Note 17 (Ref. text book page 224) 09/24/2021

3.7 Rates of Change in the Natural and Social Sciences

We know that if $y = f(x)$, then the derivative dy/dx can be interpreted as the rate of change of y with respect to x. In this section we examine some of the applications of this idea to physics, chemistry, biology, economics, and other sciences.

Let's recall from Section 2.7 the basic idea behind rates of change, if x changes from x_1 to x_2, then the change in x is $\Delta x = x_2 - x_1$ and the corresponding change in y is $\Delta y = f(x_2) - f(x_1)$. So the difference quotient $\dfrac{\Delta y}{\Delta x} = \dfrac{f(x_2) - f(x_1)}{x_2 - x_1}$ is **the average rate of change of y with respect to x** over the interval $[x_1, x_2]$. Its limit as $\Delta x \to 0$ is the derivative $f'(x_1)$, which can therefore be interpreted as the **instantaneous rate of change of y with respect to x** or the slope of the tangent line at $P(x_1, f(x_1))$. We write the process in the form

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

Whenever the function $y = f(x)$ has a specific interpretation in one of the sciences, its derivative will have a specific interpretation as a rate of change.

Physics

As we discussed in Sections 2.7 and 2.8, if $s = f(t)$ is the position function of a particle that is moving in a straight line then, $v = ds/dt$ is the instantaneous **velocity** and the **acceleration** is the rate of change of velocity i.e. $a(t) = v'(t)$.

Example 1 The position of a particle is given by the equation $s = f(t) = t^3 - 6t^2 + 9t$ where t is measured in seconds and s in meters.

(a) Find the velocity at time t.

(b) What is the velocity after 2 second? 4 second?

(c) When is the particle at rest?

(d) When is the particle moving in the positive direction?

(e) Find the total distance traveled by the particle during the first 5 seconds.

(f) Find the acceleration at time t and after 4 second.

(g) When is the particle speeding up? When is it slowing down?

Example 2

If a ball is thrown vertically upward with a velocity of 128 ft/s, then its height after t seconds is $s = 128t - 16t^2$.

(a) What is the maximum height reached by the ball?

(b) What is the velocity of the ball when it is 240 ft above the ground on its way up? (Consider up to be the positive direction.)

What is the velocity of the ball when it is 240 ft above the ground on its way down?

Chemistry

A chemical reaction results in the formation of one or more substances (called products) from one or more starting materials (called reactants). For instance, the "equation"

$$2H_2 + O_2 \longrightarrow 2H_2O$$

indicates that two molecules of hydrogen and one molecule of oxygen form two molecules of water. Let's consider the reaction

$$A + B \longrightarrow C$$

where A and B are the reactants and C is the product. The concentration of a reactant A is the number of moles (1 mole $= 6.022 \times 10^{23}$ molecules) per liter and is denoted by $[A]$. The concentration varies during a reaction, so $[A]$, $[B]$, and$[C]$ are all functions of time (t). The average rate of reaction of the product C over a time interval $t_1 \leq t \leq t_2$ is

$$\frac{\Delta C}{\Delta t} = \frac{[C](t_2) - [C](t_1)}{t_2 - t_1}$$

But chemists are more interested in the instantaneous rate of reaction, which is obtained by taking the limit of the average rate of reaction as the time interval $\triangle t$ approaches 0:

$$Rate\ of\ reaction = \lim_{\Delta t \to 0} \frac{\Delta C}{\Delta t} = \frac{dC}{dt}$$

Since the concentration of the product increases as the reaction proceeds, the derivative $\frac{dC}{dt}$ will be positive, and so the rate of reaction of C is positive. The concentrations of the reactants, however, decrease during the reaction, so, to make the rates of reaction of A and B positive numbers, we put minus signs in front of the derivatives $\frac{dA}{dt}$ and $\frac{dB}{dt}$. Since $[A]$ and $[B]$ each decrease at the same rate that $[C]$ increases, we have

$$Rate\ of\ reaction = \frac{dC}{dt} = -\frac{dA}{dt} = -\frac{dB}{dt}$$

Example 3 One molecule of the product C is formed from one molecule of the reactant A and one molecule of the reactant B, and the initial concentrations of A and B have a common value $[A] = [B] = a$ moles/L, then $[C] = \frac{a^2kt}{(akt+1)}$ where k is a constant.

(a) Find the rate of reaction at time t.

(b) Show that if $x = [C]$, then $\frac{dx}{dt} = k(a - x)^2$

(c) What happens to the concentration as $t \to \infty$?

(d) What happens to the rate of reaction as $t \to \infty$?

(e) What do the results of parts (c) and (d) mean in practical terms?

Biology

Let $n = f(t)$ be the number of individuals in an animal or plant population at time t. The change in the population size between the times $t = t_1$ and $t = t_2$ is $\Delta n = f(t_2) - f(t_1)$, and so the average rate of growth during the time period $t_1 \le t \le t_2$ is

$$\text{average rate of growth } = \frac{\Delta n}{\Delta t} = \frac{f(t_2) - f(t_1)}{t_2 - t_1}$$

The instantaneous rate of growth is obtained from this average rate of growth by letting the time period Δt approach 0:

$$\text{growth rate } = \lim_{\Delta t \to 0} \frac{\Delta n}{\Delta t} = f^{'}(t)$$

Example 4 The number of yeast cells in a laboratory culture increases rapidly initially but levels off eventually. The population is modeled by the function

$$n = f(t) = \frac{a}{1 + be^{-0.7t}}$$

where t is measured in hours. At time $t = 0$ the population is 20 cells and is increasing at a rateof 12 cells/hour. Find the values of a and b. According to this model, what happens to the yeast population in the long run?

To be more specific, consider a population of bacteria in a homogeneous nutrient medium. Suppose that by sampling the population at certain intervals it is determined that the population doubles every hour. Then the population function is $f(t) = n_0 2^t$, n_0 is the initial population.

Economics

Suppose $C(x)$ is the total cost that a company incurs in producing x units of a certain cmmodity. The function C is called a **cost function**. If the number of items produced is increased from x_1 to x_2, then the aditional cost is $\Delta C = C(x_2) - C(x_1)$, and the average rate of change of the cost is

$$\frac{\Delta C}{\Delta x} = \frac{C(x_2) - C(x_1)}{x_2 - x_1}$$

The limit of this quantity as $\Delta x \to 0$, that is, the instantaneous rate of change cost with respect to the number of items produced, is called the **marginal cost** by economists:

$$\text{marginal cost } = \lim_{\Delta x \to 0} \frac{\Delta C}{\Delta x}$$

The marginal cost of producing n units is approximately equal to the cost of producing one more unit [(n+1)st unit].

Example 5 Suppose that the cost (in dollars) for a company to produce x pairs of a new line of jeans is

$$C(x) = 2000 + 3x + 0.01x^2 + 0.0002x^3$$

(a) Find the marginal cost function.

(b) Find $C^{'}(100)$ and explain its meaning. What does it predict?

(c) Compare $C^{'}(100)$ with the cost of manufacturing the 101st pair of jeans.