4) $y(t) = y(0)e^{kt}$, $y(t)$ is the bacteria count at time t hours.

Count was 400 after 2 hours $\Rightarrow$ $y(2) = 400$

" 25,600 " 6 " $\Rightarrow$ $y(6) = 25,600$

Thus,

$$y(0)e^{2k} = 400 \quad \text{and} \quad y(0)e^{6k} = 25,600$$

$$\Rightarrow \frac{y(0)e^{6k}}{y(0)e^{2k}} = \frac{25600}{400}$$

$$\Rightarrow e^{4k} = 64$$

$$\Rightarrow 4k = \ln 64$$

$$k = \frac{1}{4}\ln 64 = \frac{1}{4}\ln 2^6 = \frac{6}{4}\ln 2 = \frac{3}{2}\ln 2$$

$$= \ln\sqrt{8}$$

a) So the relative growth rate $k = \ln\sqrt{8}$

$$\simeq 1.0397$$

i.e., about 104 % per hour.

b) Initial size $y(0) = ?$

But $y(0)e^{2k} = 400$

$$\Rightarrow y(0) = 400\, e^{-2k}$$

$$= 400\, e^{-2\ln\sqrt{8}}$$

$$= 400e^{-\ln 8}$$

$$= 400e^{\ln \frac{1}{8}}$$

$$= 400\left(\frac{1}{8}\right)$$

$$= 50$$

So the initial bacteria count was 50.

c) After t hours,

$$y(t) = 50e^{\ln\sqrt{8}\,t}$$

$$= 50\left(e^{\ln 2^{3/2}}\right)^t$$

$$= 50\left(2^{3/2 t}\right)$$

d) When $t = 4.5$ hours,

$$y(4.5) = 50\left(2^{1.5(4.5)}\right)$$

$$= 50\left(2^{6.75}\right)$$

$$\simeq 5382 \text{ bacteria}$$

e) Rate of growth after $t = 4.5$ hours $y'(4.5) = ?$

$$y(t) = 50e^{\ln\sqrt{8}\,t}$$

$$y(t) = 50e$$

$$\Rightarrow \quad y'(t) = 50\ln\sqrt{8}\, e^{\ln\sqrt{8}\,t}$$

$$= \ln\sqrt{8}\left(50e^{\ln\sqrt{8}\,t}\right)$$

$$= \ln\sqrt{8}\; y(t)$$

So

$$y'(4.5) = \ln\sqrt{8}\cdot y(4.5)$$

$$= \ln\sqrt{8}\left(50\left(2^{6.75}\right)\right)$$

$$\simeq 5596 \text{ bacteria/hour}$$

(f) Find t such that $y(t) = 50{,}000$

$$\Rightarrow \quad 50\left(2^{1.5t}\right) = 50{,}000$$

$$\Rightarrow \quad 2^{1.5t} = \frac{50{,}000}{50} = 1000$$

$$\Rightarrow \quad \ln 2^{1.5t} = \ln 1000$$

$$\Rightarrow \quad 1.5t\ln 2 = 3\ln 10$$

$$\Rightarrow \quad t = \frac{2\ln 10}{\ln 2}$$

$\simeq 6.64$ hours

Hence, the population will reach 50,000 at approximately 6.64 hours.

(5) Let $y(t)$ be the mass (in mg) remaining after t years.
Then

$$y(t) = y(0)e^{kt}.$$

$$\Rightarrow y(30) = 100e^{30k} = \tfrac{1}{2}(100) = 50$$

$$\Rightarrow 100e^{30k} = 50$$

$$\Rightarrow e^{30k} = \frac{50}{100} = 0.5$$

$$\Rightarrow \ln e^{30k} = \ln 0.5$$

$$\Rightarrow 30k \ln e = \ln 0.5$$

$$\Rightarrow 30k = \ln 0.5$$

$$\Rightarrow k = \frac{1}{30}\ln 0.5 = \frac{1}{30}\ln 2^{-1} = \frac{-\ln 2}{30}$$

(a) So, the mass that remains after t years is

$$y(t) = 100e^{-\frac{\ln 2}{30}t}$$

(b) After 100 years,

$$y(100) = 100e^{-\frac{\ln 2}{30}\cdot 100} \simeq 9.92\,\text{mg}$$

(c) Find t such that $y(t) = 1$

$$\Rightarrow \quad 100e^{-\frac{\ln 2}{30}t} = 1$$

$$\Rightarrow \quad e^{-\frac{\ln 2}{30}t} = \frac{1}{100} = 0.01$$

$$\Rightarrow \quad \ln e^{-\frac{\ln 2}{30}t} = \ln 0.01$$

$$\Rightarrow \quad -\frac{\ln 2}{30}t \ln e = \ln 0.01$$

$$\Rightarrow \quad -\frac{\ln 2}{30}t = \ln 0.01$$

$$\Rightarrow \quad t = \frac{30 \ln 0.01}{-\ln 2}$$

$$\simeq 199.3157$$

Hence, After (roughly) 199.3 years, only 1 mg of cesium-137 will remain