Lecture Note 18 (Ref. text book page 237)

09/29/2021

3.8 Exponential Growth and Decay

In many natural phenomena, quantities grow or decay at a rate proportional to their size. If $y(t)$ is the value of a quantitiy y at time t and if the rate of change of y with respect to t is propotional to its size $y(t)$ at any time, then

$$\frac{dy}{dt} = ky$$

where k is a constant and dy/dt is the growth rate.

Theorem The only solutions of the differential equation $dy/dt = ky$ are exponential functions

$$y(t) = y(0)e^{kt}$$

Population Growth

Example 1 A cell of a bacteria culture in a nutrient-broth medium divides in two cells every 20 minutes. The initial population of a culture is 50 cells.

(a) Find the relative growth-rate.

(b) Find an expression for the number of cells after t hours.

(c) Find the number of cells after 6 hours.

(d) Find the rate of growth after 6 hours.

(e) When will the population reach a million cells?

Radioactive Decay

Radioactive substances decay by spontaneously emitting radiation. Scientists express the rate of decay in terms of **half-life**, the time required for half of any given quantity to decay.

Example 2 The half-life of cesium-137 is 30 years. Suppose we have 100-mg sample.

(a) Find the mass that remains after t years.

(b) How much of the sample remains after 100 years?

(c) After how long will only 1-mg remain?

Newton's Law of Cooling

Newton's Law of Cooling states that the rate of cooling of an object is propotional to the temperature difference between the object and its surroundings, provided that this difference is not too large. (This law also applies to warming.) If we let $T(t)$ be the temprature of the object at time t and T_s be the temprature of the surrondings, then we can formulate Newton's Law of Cooling as a differential equation:

$$\frac{dT}{dt} = k(T - T_s)$$

Example 3 A roast turkey is taken from an oven when its teprature has reached 185°F and is placed on a table in a room where the temprature is 75°F.

(a) If the temprature of the turkey is 150° after half an hour, what is the temprature after 45 minutes?

(b) When will the turkey have cooled to 100°F?

Continuously Compounded Interest

Compound interest is calculated by the formula $A(t) = P\left(1+\dfrac{r}{m}\right)^{mt}$ and

Continuously compounded interest is calculated by the formula

$$A(t) = Pe^{rt}\ , \quad \text{where}$$

$A(t) =$ amount after t years,
$P =$ principal,
$r =$ interest rate, (nominal) per year,
$m =$ number of times interested is compounded per year,
$t =$ number of years.

Example 4

(a) If \$3000 is invested at 5% interest, find the value of the investment at the end of 5 years if the interest is compunded (i) annualy, (ii) semiannually, (iii) monthly, (iv) weekly, (v) daily, and (vi) continuously.

(b) If $A(t)$ is the amount of the investment at time t for the case of continuous compunding, write a differential equation and an initial condition satisfied by $A(t)$.