Lecture Note 23 (Ref. text book page 293) 10/15/2021

4.3 How Derivatives Affect the Shape of a Graph

Many of the applications of calculus depend on our ability to deduce facts about a function f from information concerning its derivatives. Because $f'(x)$ represents the slope of the curve $y = f(x)$ at the point $(x, f(x))$, it tells us the direction in which the curve proceeds at each point. So it is reasonable to expect that information about $f'(x)$ will provide us with information about $f(x)$.

Increasing/Decreasing Test

(a) If $f'(x) > 0$ on an interval, then f is increasing on that interval.

(b) If $f'(x) < 0$ on an interval, then f is decreasing on that interval.

Determining the Intervals Where a Function Is Increasing or Decreasing

1. Find all values of x for which $f'(x) = 0$ or f' is discontinuous, and identify the open intervals determined by these numbers.
2. Select a test number c in each interval found in step 1, and determine the sign of $f'(c)$ in that interval.

Example 1 Find the intervals on which $f(x) = x^4 - 2x^2 + 3$ is increasing and decreasing.

The First Derivative Test Suppose that c is a critical number of a continuous function f.

(a) If f' changes from positive to negative at c, then f has a local maximum at c.

(b) If f' changes from negative to positive at c, then f has a local minimum at c.

(c) If f' is positive to the left and right of c, or negative to the left and right of c, then f has no local maximum or minimum at c.

Example 2 Find the local maximum and minimum values of $f(x) = x^4 - 2x^2 + 3$.

Definition

1. If the graph of f lies above all of its tangents on an interval I, then it is called **concave upward** on I.
2. If the graph of f lies below all of its tangents on I, it is called **concave downward** on I.
3. A point P on a curve $y = f(x)$ is called an **inflection point** if f is continuous there and the curve changes from concave upward to concave downward or from concave downward to concave upward at P.

Concavity Test

(a) If $f''(x) > 0$ for all x in I, then the graph of f is concave upward on I.

(b) If $f''(x) < 0$ for all x in I, then the graph of f is concave downward on I.

Example 3 Find the intervals of concavity and the inflection points. $f(x) = x^4 - 2x^2 + 3$.

Example 4 For function $f(x) = \cos^2 x - 2\sin x, 0 \leq x \leq 2\pi$

(a) Find the intervals on which f is increasing or decreasing.

(b) Find the local maximum and minimum values of f.

(c) Find the intervals of concavity and the inflection points.

Example 5 Sketch the graph of a function that satisfies all of the given conditions.

$f'(5) = 0$, $f'(x) < 0$ when $x < 5$, $f'(x) > 0$ when $x > 5$, $f''(2) = 0$, $f''(8) = 0$, $f''(x) < 0$ when $x < 2$ or $x > 8$, $f''(x) > 0$ for $2 < x < 8$, $\lim_{x\to\infty} f(x) = 3$, $\lim_{x\to-\infty} f(x) = 3$

The Second Derivative Test Suppose f'' is continuous near c.

(a) If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at c.

(b) If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at c.

NOTE The Second Derivative Test is inconclusive when $f''(c) = 0$. In other words, at such a point there might be a maximum , there might be a minimum, or there might be neither. This test also fails when $f''(c)$ does not exist. In such cases the First Derivative Test must be used. In fact, even when both tests apply, the First Derivative Test is often the easier one to use.

More Examples on 4.2

19–20 Show that the equation has exactly one real root.

19. $2x + \cos x = 0$

20. $x^3 + e^x = 0$

21. Show that the equation $x^3 - 15x + c = 0$ has at most one root in the interval $[-2, 2]$.

25. If $f(1) = 10$ and $f'(x) \geqslant 2$ for $1 \leqslant x \leqslant 4$, how small can $f(4)$ possibly be?