Lecture Note 24 (Ref. text book page 304) 10/18/2021

4.4 Indeterminate Forms and l'Hospital's Rule

The Indeterminate Forms $0/0$ and ∞/∞

1. If $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$, then the limit $\lim_{x\to a} \dfrac{f(x)}{g(x)}$ is called an **indeterminate form of** $0/0$.

2. If $\lim_{x\to a} f(x) = \pm\infty$ and $\lim_{x\to a} g(x) = \pm\infty$, then the limit $\lim_{x\to a} \dfrac{f(x)}{g(x)}$ is also an indeterminate form of the type ∞/∞, $-\infty/\infty$, $\infty/-\infty$, or $-\infty/-\infty$. We refer to each of these limits as an **indeterminate form of the type** ∞/∞, since the sign provides little useful information.

l'Hospital's Rule

Suppose f and g are differentiable on an open interval I that contains a, with the possible exception of a itself and $g'(x) \neq 0$ for all x in I, with the possible exception of a. If $\lim_{x\to a} \dfrac{f(x)}{g(x)}$ has an indeterminate form of the type $0/0$ or ∞/∞, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

provided that the limit on the right exists or is infinite.

Notes

1. l'Hospital's Rule is valid for one-sided limits as well as limits at infinity or negative infinity; that is, we can replace $x \to a$ by any of the symbols $x \to a^+$, $x \to a^-$, $x \to -\infty$, $x \to \infty$.

2. Before applying l'Hospital's Rule, check to see that the limit has one of the indeterminate forms. For example
$$\lim_{x\to 0^+} \frac{1}{x} = \infty$$
If we had applied l'Hospital's Rule to evaluate the limit without first ascertaining that it had an indeterminate form, we would have obtained the erroneous result
$$\lim_{x\to 0^+} \frac{1}{x} = \lim_{x\to 0^+} \frac{0}{1} = 0$$

Example 1 Evaluate each limit. (Show work, if DNE, must prove.)

(a) $\lim_{x\to 1} \dfrac{\ln x}{x-1}$ (b) $\lim_{x\to 0} \dfrac{e^x - 1 - x}{x^2}$ (c) $\lim_{x\to -2} \dfrac{x^3+8}{x+2}$ (d) $\lim_{x\to\infty} \dfrac{e^x}{x^2}$.

Indeterminate Products

If $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = \infty$ or $(-\infty)$, then it isn't clear what the value of $\lim_{x\to a}[f(x)g(x)]$, if any, will be. This kind of limit is called an **indeterminate form of type 0.**∞. We can deal with it by writing the product fg as a quotient

$$fg = \frac{f}{1/g} \quad \text{or} \quad fg = \frac{g}{1/f}$$

This converts the given limit into an indeterminate form of type $0/0$ or ∞/∞ so that we can use l'Hospital's Rule.

Indeterminate Differences

If $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = \infty$, then the limit $\lim_{x\to a}[f(x) - g(x)]$ is called an **indeterminate form of type** $\infty - \infty$. To deal with such a limit, we try to convert the difference into a quotient (for instance, by using a common denominator, or rationalization, or factoring out a common factor) so that we have an indeterminate form of type $0/0$ or ∞/∞ so that we can use l'Hospital's Rule.

Example 2 Evaluate each limit. (Show work, if DNE, must prove.)

(a) $\lim\limits_{x\to\infty} x\sin(\pi/x)$

(b) $\lim\limits_{x\to 1} \left(\dfrac{x}{x-1} - \dfrac{1}{\ln x}\right)$

Indeterminate Powers

Several indeterminate forms arise from the limit

$$\lim_{x\to a}[f(x)]^{g(x)}$$

1. $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$ type 0^0
2. $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = 0$ type ∞^0
3. $\lim_{x\to a} f(x) = 1$ and $\lim_{x\to a} g(x) = \pm\infty$ type 1^∞

Each of these three cases can be treated by taking the natural logarithm:

$$\text{let} \quad y = [f(x)]^{g(x)}, \quad \text{then} \quad \ln y = g(x) \ln f(x)$$

In this method we are led to the indeterminate product $g(x)\ln f(x)$, which is of type $0 \cdot \infty$.

Example 3 Evaluate each limit. (Show work, if DNE, must prove.)

(a) $\displaystyle\lim_{x\to\infty}\left(1+\frac{a}{x}\right)^{bx}$

(b) $\displaystyle\lim_{x\to\infty} x^{e^{-x}}$