Wednesday, October 20, 2021 10:08 AM

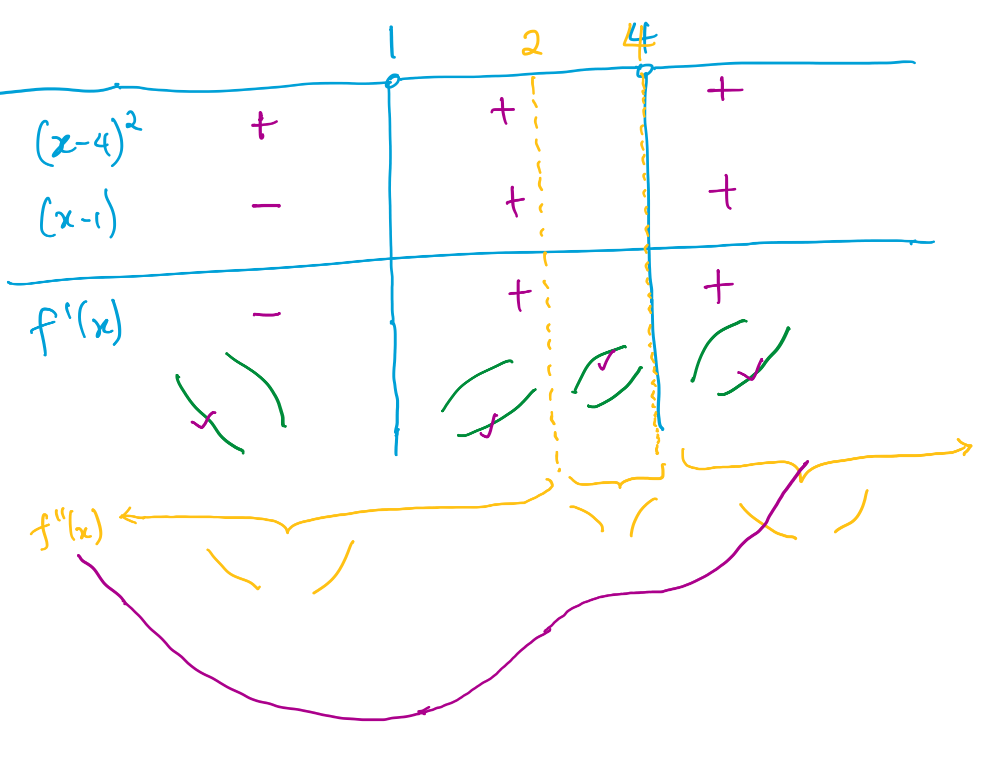
<u>Example 1</u>: Use the guidelines to sketch the following curve.

(a) $y = x(x-4)^3$.

$\mathrm{Dom}(y) = (-\infty, \infty)$.

$$f'(x) = (x-4)^3 + 3x(x-4)^2$$
$$= (x-4)^2\left[x-4+3x\right]$$
$$= (x-4)^2(4x-4)$$
$$= 4(x-4)^2(x-1)$$

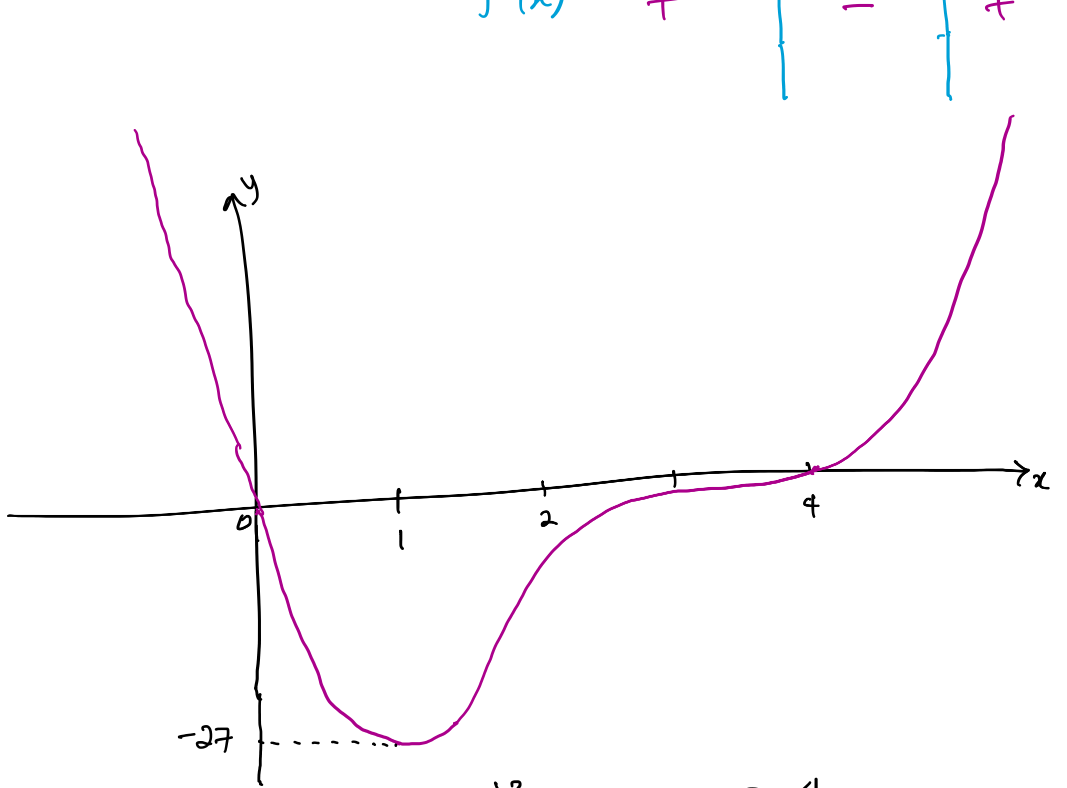
$\Rightarrow$ Critical numbers are $x = 1, 4$.



$$f''(x) = 4\times 2(x-4)(x-1) + 4(x-4)^2$$
$$= 4(x-4)\left[2x-2+x-4\right]$$
$$= 4(x-4)(3x-6)$$
$$= 12(x-4)(x-2)$$

$\Rightarrow x = 2, 4$

		2		4	
$x-4$	$-$		$-$		$+$
$x-2$	$-$		$+$		$+$
$f''(x)$	$+$		$-$		$+$



x-intercepts: $y = 0 \Rightarrow x(x-4)^3 = 0 \Rightarrow x = 0, 4$
i.e. $(0,0)$, $(4,0)$ on the graph.

y-intercept: At $x=0$, $y=0$.
i.e. $(0,0)$ on the graph

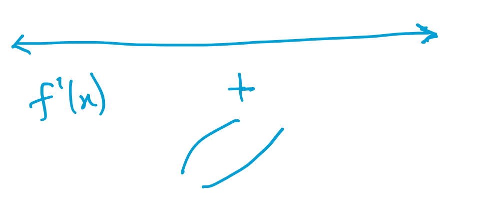
$$\lim_{x\to-\infty} y = \lim_{x\to-\infty} x(x-4)^3 = \infty$$
$$\lim_{x\to\infty} y = \lim_{x\to\infty} x(x-4)^3 = \infty$$

local min value $= f(1) = 1(1-4)^3 = -27$

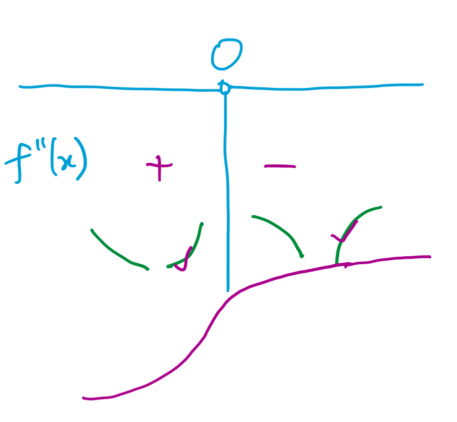
(b) $y = \dfrac{x}{\sqrt{x^2+1}} = \dfrac{x}{(x^2+1)^{1/2}}$

Domain $= \mathbb{R}$.

$$y'(x) = \frac{(x^2+1)^{1/2} - x\cdot\frac{1}{2}(x^2+1)^{-1/2}(2x)}{\left[(x^2+1)^{1/2}\right]^2}$$
$$= \frac{\sqrt{x^2+1} - \frac{x^2}{\sqrt{x^2+1}}}{x^2+1} = \frac{x^2+1-x^2}{(x^2+1)\sqrt{x^2+1}}$$
$$= \frac{1}{(x^2+1)^{3/2}} = (x^2+1)^{-3/2}$$

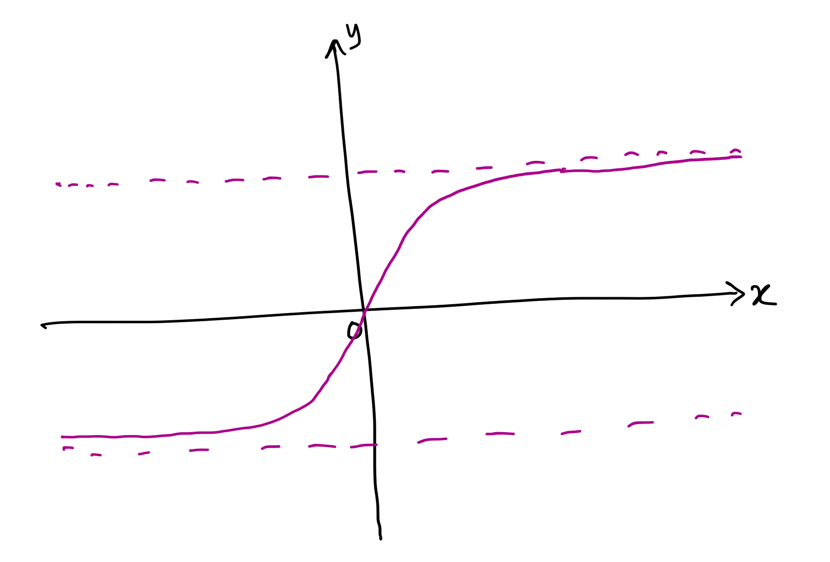


$$y''(x) = -\frac{3}{2}(x^2+1)^{-5/2}(2x) = \frac{-3x}{(x^2+1)^{5/2}} = 0 \text{ if } x=0.$$



Also,
$$\lim_{x\to\infty}\frac{x}{\sqrt{x^2+1}} = \lim_{x\to\infty}\frac{\cancel{x}(1)}{\cancel{x}\sqrt{1+\frac{1}{x^2}}} = \lim_{x\to\infty}\frac{1}{\sqrt{1+\frac{1}{x^2}}} = 1$$
$$\lim_{x\to-\infty}\frac{x}{\sqrt{x^2+1}} = \lim_{x\to-\infty}\frac{x(1)}{|x|\sqrt{1+\frac{1}{x^2}}} = \lim_{x\to-\infty}\frac{-1}{\sqrt{1+\frac{1}{x^2}}} = -1$$

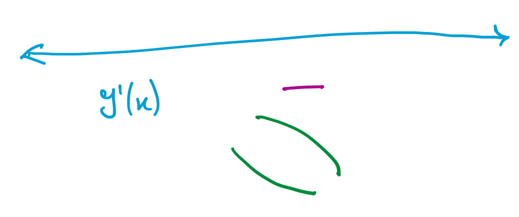
So $y = 1$ and $y = -1$ are horizontal asymptotes.



(c) $y = (1+e^x)^{-2} = \dfrac{1}{(1+e^x)^2}$

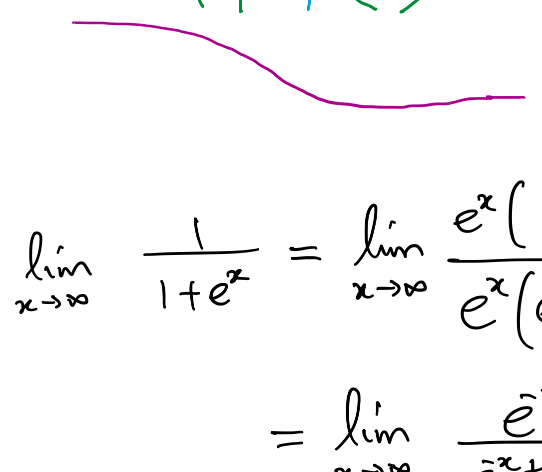
Domain = all real numbers.

$$y'(x) = -2(1+e^x)^{-3}(e^x) = \frac{-2e^x}{(1+e^x)^3}$$



$$y''(x) = \frac{(1+e^x)^3(-2e^x) + 2e^x(3e^x)(1+e^x)^2}{(1+e^x)^6}$$
$$= \frac{(1+e^x)^2\left[-2e^x(1+e^x) + 6e^{2x}\right]}{(1+e^x)^6}$$
$$= \frac{(1+e^x)^2\left[4e^{2x} - 2e^x\right]}{(1+e^x)^6}$$

So $y''(x) = 0 \Rightarrow 4e^{2x} - 2e^x = 0$
$\Rightarrow 4e^{2x} = 2e^x$
$\Rightarrow e^{2x-x} = \frac{2}{4}$
$\Rightarrow e^x = \frac{2}{4}$
$\Rightarrow x = \ln(2/4) = \ln(1/2)$



Also,
$$\lim_{x\to\infty}(1+e^x)^2 = \lim_{x\to\infty}\frac{1}{1+e^x} = \lim_{x\to\infty}\frac{e^x(e^{-x})}{e^x(e^{-x}+1)}$$
$$= \lim_{x\to\infty}\frac{e^{-x}}{e^{-x}+1} = \frac{0}{0+1} = 0$$

and
$$\lim_{x\to-\infty}(1+e^x)^2 = \lim_{x\to-\infty}\frac{1}{1+e^x} = \frac{1}{1+0} = 1$$

So $y = 0$ and $y = 1$ are horizontal asymptotes.