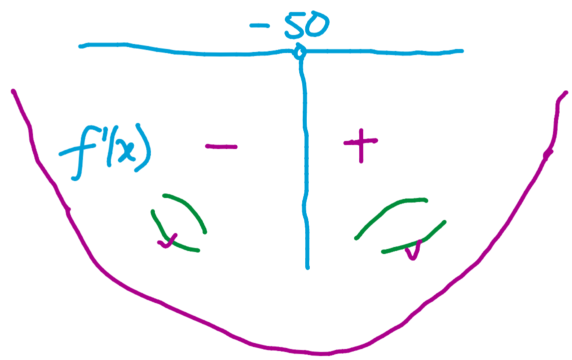
Example 1 Find two numbers whose **difference** is 100 and whose product is a minimum.

Let x and y be the numbers. Then $\underbrace{y - x = 100}_{y = 100 + x}$. So, we seek to find the minimum of

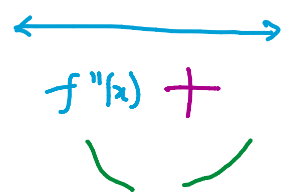
$$f(x) = xy$$
$$= x(100 + x)$$
$$= 100x + x^2$$



$$f'(x) = 100 + 2x = 0$$
$$\Rightarrow x = -50$$

Also,

$$f''(x) = 2.$$



From the sign tables, we see that f attains minimum value at $x = -50$.

Hence, the two numbers are -50 and 50.

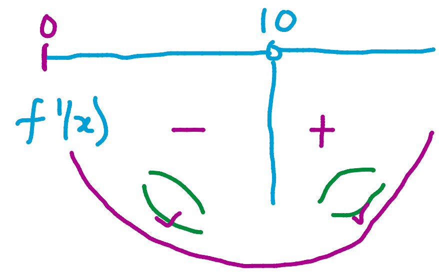
Example 2 Find two positive numbers whose product is 100 and whose sum is a minimum.

Let the numbers be x and y. Then $xy = 100 \Rightarrow y = \frac{100}{x}$ and we seek to minimize

$$f(x) = x + y$$
$$= x + \frac{100}{x}$$

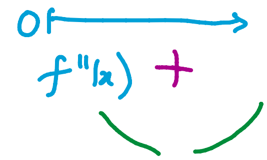
$$f'(x) = 1 - \frac{100}{x^2} = \begin{cases} 0 & \text{if } x = -10, 10 \\ -\infty & \text{if } x = 0 \end{cases}$$

Since the numbers are positive, we only consider $x=10$ as potentially the the number at which f is minimum



Also,

$$f''(x) = \frac{200}{x^3} > 0 \text{ for } x > 0$$

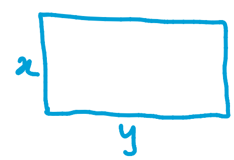


From the sign tables, we clearly see that the sum is minimum at $x = 10$. So $y = \frac{100}{10} = 10$.

Hence, the two numbers 10 and 10.

Example 3 Find the dimensions of a rectangle with perimeter $100m$ whose area is as large as possible.

Let x be the width and the y, the length of the rectangle.



Then $2x + 2y = 100 \Rightarrow y = 50 - x$.

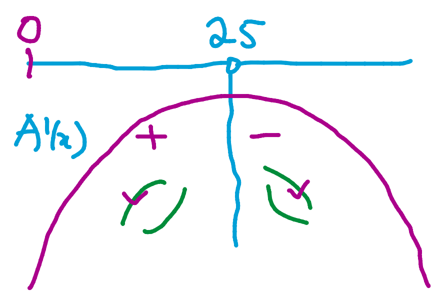
So

area $A(x) = xy$

$$= x(50-x)$$

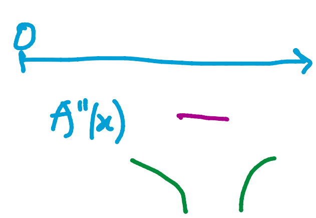
$$= 50x - x^2$$

$$A'(x) = 50 - 2x = 0 \Rightarrow x = 25$$



Also,

$$A''(x) = -2.$$

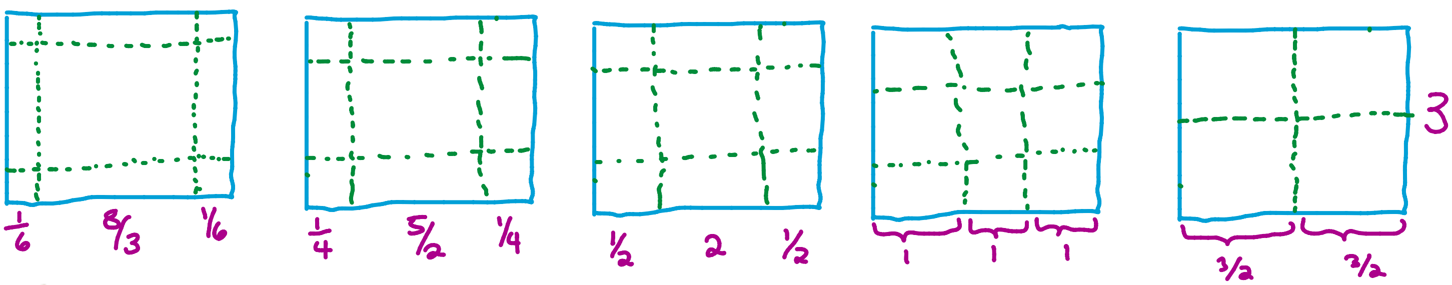


From the sign tables, we see that f is largest at $x = 25$. So $y = 50 - x = 50 - 25 = 25$

Hence, the dimension of the rectangle is 25m by 25m. i.e; a square.

Example 4 Consider the following problem: A box with an open top is to be constructed from a square piece of cardboard, 3 ft wide, by cutting out a square from each of the four corners and bending up the sides. Find the largest volume that such a box can have.

(a) Draw several diagrams to illustrate the situation, some short boxes with large bases and some tall boxes with small bases. Find the volumes of several such boxes. Does it appear that there is a maximum volume? If so, estimate it.



Volume $= \frac{8}{3} \times \frac{8}{3} \times \frac{1}{6} = \frac{32}{27}$

$V = \frac{5}{2} \times \frac{5}{2} \times \frac{1}{4} = \frac{25}{16}$

$V = 2 \times 2 \times \frac{1}{2} = 2$

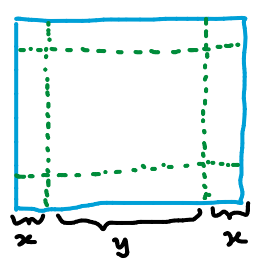
$V = 1 \times 1 \times 1 = 1$

$V = 0 \times 0 \times \frac{3}{2} = 0$

For these values, there appear to be maximum volume of 2.

(b) Draw a diagram illustrating the general situation. Introduce notation and label the diagram with your symbols.

Consider the general setting as shown. Then $0 \le x \le 3/2$ and



(c) Write an expression for the volume.

Volume V = area of base × height

$= y \times y \times x$

$= xy^2.$

(d) Use the given information to write an equation that relates the variables.

Given that length of the cardboard is 3 ft,

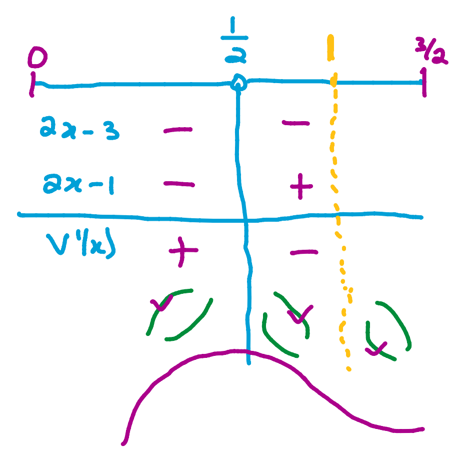
$$x + x + y = 3 \Rightarrow y = 3 - 2x.$$

(e) Use part (d) to write the volume as a function of one variable.

$$\begin{aligned} V(x) &= xy^2 \\ &= x(3-2x)^2 \\ &= x(9 - 12x + 4x^2) \\ &= 9x - 12x^2 + 4x^3. \end{aligned}$$

(f) Finish solving the problem and compare the answer with your estimate in part (a).

$$\begin{aligned} V'(x) &= 9 - 24x + 12x^2 = 0 \\ &\Rightarrow 4x^2 - 8x + 3 = 0 \\ &\Rightarrow 4x^2 - 2x - 6x + 3 = 0 \\ &\Rightarrow 2x(2x-1) - 3(2x-1) = 0 \\ &\Rightarrow (2x-3)(2x-1) = 0 \\ &\Rightarrow x = \tfrac{1}{2}, \tfrac{3}{2}. \end{aligned}$$



Also,

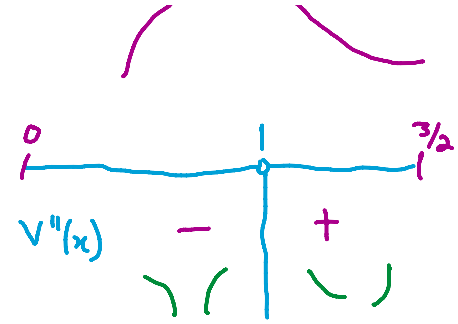
$$V''(x) = -24 + 24x = 0$$

$$\Rightarrow x = 1$$

From the sign tables, V attains maximum at $x = \frac{1}{2}$.

From the sign charts, V attains maximum at $x = 1/2$.

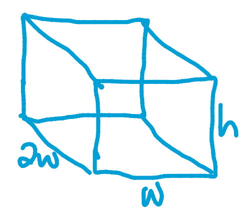
Luckily, this result coincides with our estimates in part (a).



Example 5 A rectangular storage container with an open top is to have a volume of $10m^3$. The length of its base is twice the width. Material for the base costs \$10 per square meter. Material for the sides costs \$6 per square meter. Find the cost of materials for the cheapest such container.

Volume

$$V = lwh = 2w(w)h = 2w^2h$$



So,

$$2w^2h = 10 \Rightarrow h = \frac{5}{w^2}$$

Let C be the cost function associated with constructing the surfaces of the container. Then

$$C = \text{cost of base} + \text{cost of longer rectangular faces} + \text{cost of shorter rectangular faces}$$

$$= 10(2w^2) + 6[2(2wh)] + 6[2wh].$$

$$= 20w^2 + 24w\left(\frac{5}{w^2}\right) + 12w\left(\frac{5}{w^2}\right)$$

$$= 20w^2 + \frac{120}{w} + \frac{60}{w}$$

$$C(w) = 20w^2 + \frac{180}{w}.$$

Cheapest cost means find the minimum value of the cost.

$$C'(w) = 40w - \frac{180}{w^2} = 0 \Rightarrow 40w^3 = 180 \Rightarrow w = \sqrt[3]{\frac{180}{40}} = \sqrt[3]{\frac{25}{16}}$$

Also,

$$C''(x) = 40 + \frac{360}{w^3} > 0 \text{ since } w > 0. \left(\text{so } C(x) \text{ is concave up}\right).$$

Hence, the cheapest such container is constructed with the width of $\sqrt[3]{\frac{25}{16}}$ ft.

Example 6 Find the points on the ellipse $4x^2 + y^2 = 4$ that are farthest away from the point $(1, 0)$.

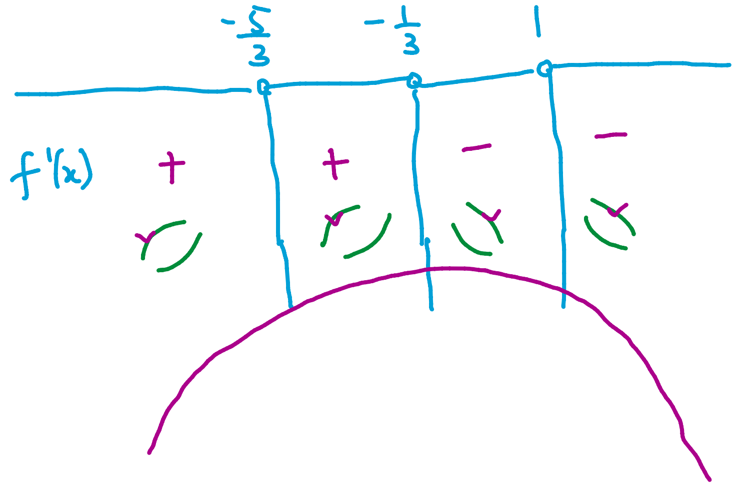
i.e; Maximize $f = \sqrt{(x-1)^2 + (y-0)^2}$ subject to $4x^2+y^2=4$.

$$
\begin{aligned}
&= \sqrt{(x-1)^2 + y^2} \\
&= \sqrt{(x-1)^2 + 4 - 4x^2} \\
&= \sqrt{x^2 - 2x + 1 + 4 - 4x^2} \\
&= \sqrt{5 - 2x - 3x^2} \\
&= (5 - 2x - 3x^2)^{1/2}
\end{aligned}
$$

$$
\begin{aligned}
f'(x) &= \frac{1}{2}(5 - 2x - 3x^2)^{-1/2}(-2 - 6x) \\
&= \frac{-1-3x}{\sqrt{5-2x-3x^2}} \\
&= \frac{-1-3x}{\sqrt{(3x+5)(1-x)}} = \begin{cases} 0 & \text{if } x = -\frac{1}{3} \\ -\infty & \text{if } x = -\frac{5}{3}, 1 \end{cases}
\end{aligned}
$$

So $x = -\frac{5}{3}, -\frac{1}{3}, 1$ are potential

So $x = -\frac{5}{3}, -\frac{1}{3}, 1$ are potential critical numbers.



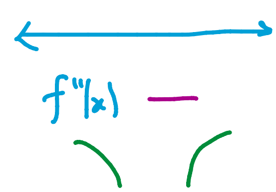
Also,

$$f''(x) = \frac{\sqrt{5-2x-3x^2}\,(-3) - (-1-3x)\dfrac{(-1-3x)}{\sqrt{5-2x-3x^2}}}{5-2x-3x^2}$$

$$= \frac{(5-2x-3x^2)(-3) - (1+3x)^2}{(5-2x-3x^2)^{3/2}}$$

$$= \frac{-15+6x+9x^2-(1+6x+9x^2)}{(5-2x-3x^2)^{3/2}}$$

$$= \frac{-16}{(5-2x-3x^2)^{3/2}} < 0 \text{ for all } x.$$



From the sign tables, we see that the maximum occurs at $x = -\frac{1}{3}$

So $4x^2+y^2 = 4 \Rightarrow 4\left(-\frac{1}{3}\right)^2 + y^2 = 4$

$$\Rightarrow \quad 4 + 9y^2 = 36$$

$$\Rightarrow \quad 9y^2 = 32$$

$$y = \pm\sqrt{\frac{32}{9}}$$

$$= \pm\frac{4\sqrt{2}}{3}$$

Hence, the points on the ellipse $4x^2+y^2=4$ that are farthest from the point $(1,0)$ are $\left(-\frac{1}{3}, \frac{4\sqrt{2}}{3}\right)$ and $\left(-\frac{1}{3}, \frac{-4\sqrt{2}}{3}\right)$.

Example 7 Find the point on the line $y = 2x + 3$ that is closest to the origin.

Mathematically, the question is asking to

Minimize $f = \sqrt{(x-0)^2+(y-0)^2}$ subject to $y = 2x+3$

$$= (x^2+y^2)^{1/2}$$

$$= \left(x^2+(2x+3)^2\right)^{1/2}$$

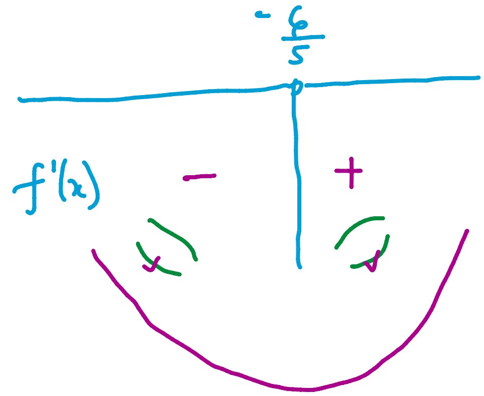
$$= \left(x^2+4x^2+12x+9\right)^{1/2}$$

$$= \left(5x^2+12x+9\right)^{1/2}$$

$$f'(x) = \frac{1}{2}\left(5x^2+12x+9\right)^{-1/2}(10x+12)$$

$$= \frac{5x+6}{\sqrt{5x^2+12x+9}}$$

$\Rightarrow \quad x = -\frac{6}{5}$ is the critical number.



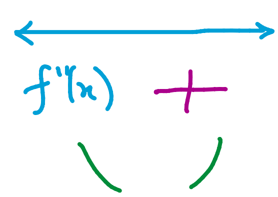
Also,

$$f''(x) = \frac{\sqrt{5x^2+12x+9}\,(5) - (5x+6)\dfrac{(5x+6)}{\sqrt{5x^2+12x+9}}}{5x^2+12x+9}$$

$$= \frac{5(5x^2+12x+9) - (5x+6)^2}{(5x^2+12x+9)^{3/2}}$$

$$= \frac{25x^2+60x+45 - (25x^2+60x+36)}{(5x^2+12x+9)^{3/2}}$$

$$= \frac{9}{(5x^2+12x+9)^{3/2}} < 0 \text{ for all } x.$$

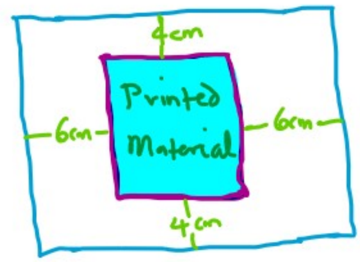


So the origin is closest to $y = 2x+3$ at $x = -6/5 \Rightarrow y = 2\left(\frac{-6}{5}\right)+3 = \frac{3}{5}$

Hence, the point on the graph of $y = 2x+3$ closest to the origin is $\left(-\frac{6}{5}, \frac{3}{5}\right)$.

$\left(-\frac{6}{5}, \frac{3}{5}\right)$.

Example 8 The top and bottom margins of a poster are each $6cm$ and the side margins are each $4cm$. If the area of printed material on the poster is fixed at $384cm^2$, find the dimensions of the poster with the smallest area.



Let y and x be the length and width of the poster respectively. Then $x-8$ and $y-12$ are the width and length of the printed material respectively.

Then

$$(x-8)(y-12) = 384$$

$$x(y-12) - 8(y-12) = 384$$

$$xy - 12x - 8y + 96 = 384$$

$$(x-8)y = 288 + 12x$$

$$y = \frac{288 + 12x}{x-8}$$

Thus, area of the poster

$$A = xy$$

$$= x\left(\frac{288 + 12x}{x-8}\right)$$

$$= \frac{288x + 12x^2}{x-8}$$

We now minimize this area:

$$A'(x) = \frac{(x-8)(288+24x) - (288x + 12x^2)(1)}{(x-8)^2}$$

$$= \frac{\cancel{288x} + 24x^2 - 2304 - 192x - \cancel{288x} - 12x^2}{(x-8)^2}$$

$$= \frac{\cancel{2}88x + 24x^2 - 2304 - 192x}{(x-8)^2}$$

$$= \frac{12x^2 - 192x - 2304}{(x-8)^2}$$

$$= \frac{12\left(x^2 - 16x - 192\right)}{(x-8)^2}$$

$$= \frac{12(x-24)(x+8)}{(x-8)^2} = \begin{cases} 0 & \text{if } x = -8, 24 \\ -\infty & \text{if } x = 8 \end{cases}$$

Since x is distance, $x > 0$.

	0	8	24
$x-24$	$-$	$-$	$+$
$x+8$	$+$	$+$	$+$
$(x-8)^2$	$+$	$+$	$+$
$A'(x)$	$-$	$-$	$+$

Also,

$$A''(x) = \left(\frac{12x^2 - 192x - 2304}{(x-8)^2}\right)'$$

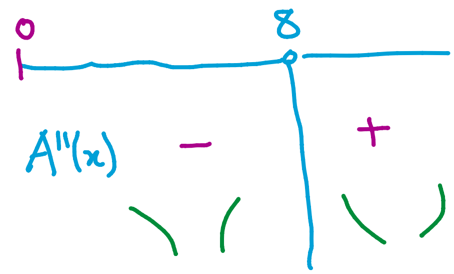
$$= \frac{(x-8)^2(24x - 192) - \left(12x^2 - 192x - 2304\right)\left(2(x-8)\right)}{(x-8)^4}$$

$$= \frac{(x-8)\left[(x-8)(24x-192) - 2\left(12x^2 - 192x - 2304\right)\right]}{}$$

$$= \frac{(x-8)\left[(x-8)(24x-192) \ \ldots \ \right]}{(x-8)^4}$$

$$= \frac{\cancel{24x^2} - \cancel{192x} - \cancel{192x} + 1536 - \cancel{24x^2} + \cancel{384x} + 4608}{(x-8)^3}$$

$$= \frac{6144}{(x-8)^3}$$



Furthermore,

$$\lim_{x\to\infty} \frac{288x + 12x^2}{x-8} = \infty, \qquad \lim_{x\to -\infty} \frac{288x+12x^2}{x-8} = \infty.$$

and $x = 8$ is a vertical asymptote. Moreover, $x > 8$, otherwise $x - 8 < 0 \Rightarrow A(x) < 0$ (Impossible!)

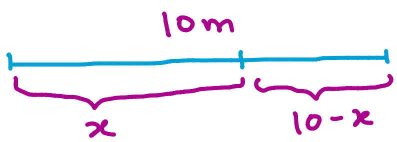
So, the smallest area occurs at $x = 24$ and $y = \frac{288 + 12(24)}{24-8} = 36$.

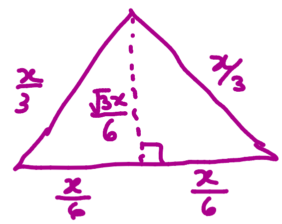
So the dimensions of the poster with the smallest area is 24cm by 36cm.

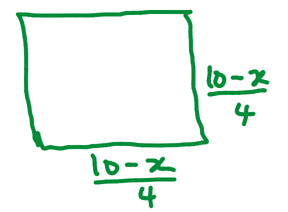
Example 9 A piece of wire 10 m long is cut into two pieces. One piece is bent into a square and the other is bent into an equilateral triangle. How should the wire be cut so that the total area enclosed is

(a) a maximum?

(b) a minimum?







Assuming the x side is used for the equilateral triangle and $10-x$ for the square

$$\text{Area} = \frac{1}{2}\left(\frac{x}{3}\right)\left(\frac{\sqrt{3}}{6}x\right)$$

$$\text{Area} = \left(\frac{10-x}{4}\right)^2$$

So total area enclosed:

$$A = \frac{1}{2}\left(\frac{x}{3}\right)\left(\frac{\sqrt{3}}{6}x\right) + \left(\frac{10-x}{4}\right)^2$$

$$A(x) = \frac{\sqrt{3}\,x^2}{36} + \frac{100 - 20x + x^2}{16}$$

Thus,

$$A'(x) = \frac{\sqrt{3}}{18}x + \frac{-20+2x}{16} = 0$$

$$\Rightarrow \quad 16\sqrt{3}\,x + 18(-20+2x) = 0$$

$$\Rightarrow \quad (16\sqrt{3} + 36)x = 360$$

$$\Rightarrow \quad x = \frac{360}{16\sqrt{3}+36} = \frac{90}{4\sqrt{3}+9} \approx 5.65$$

Clearly $0 \le x \le 10$ where $x = 10$ corresponds when the wire is used for

Clearly $0 \le x \le 10$ where $x = 10$ corresponds when the wire is used for the equilateral triangle and $x=0$ corresponds to when the whole wire is used for the square. So,

$$A(0) = \frac{100}{16} = 6.25$$

$$A\left(\frac{90}{4\sqrt{3}+9}\right) \approx 2.7185$$

$$A(10) \approx 4.8113$$

Hence, the total area enclosed is

ⓐ maximum when the wire is not cut at all and is used for a square.

ⓑ minimum when ≈ 5.65 of the wire is used to make an equilateral triangle