⑨ $\int_0^8 \sin(\sqrt{x})\,dx$, $n=4$ $\iff$ Estimate the area under the curve $f(x) = \sin\sqrt{x}$ from $x=0$ to $x=2$ using 4 rectangles and mid-points.

By the midpoint rule,

$$\int_0^8 \sin(\sqrt{x})dx \approx \sum_{i=1}^{4} f(\bar{x}_i)\Delta x$$
$$= \left[f(\bar{x}_1) + f(\bar{x}_2) + f(\bar{x}_3) + f(\bar{x}_4)\right]\cdot \Delta x$$

But
$$\Delta x = \frac{8-0}{4} = 2$$

So
$$\bar{x}_1 = 1,\ \bar{x}_2 = 3,\ \bar{x}_3 = 5,\ \bar{x}_4 = 7.$$

Thus

$$\int_0^8 \sin(\sqrt{x})dx \approx \sum_{i=1}^{4} f(\bar{x}_i)\Delta x$$
$$= \left[f(\bar{x}_1) + f(\bar{x}_2) + f(\bar{x}_3) + f(\bar{x}_4)\right]\cdot \Delta x$$
$$= \left[f(1) + f(3) + f(5) + f(7)\right]\cdot 2$$
$$= \left(\sin\sqrt{1} + \sin\sqrt{3} + \sin\sqrt{5} + \sin\sqrt{7}\right)\cdot 2$$
$$\approx (3.0910)\cdot 2 = 6.1820$$

(10) $\int_0^1 \sqrt{x^3+1}\,dx$, $n=5$

By the Mid-point rule,

$$\int_0^1 \sqrt{x^3+1}\,dx \approx \sum_{i=1}^{5} f(\bar{x}_i)\Delta x \quad \text{where } f(x)=\sqrt{x^3+1}$$

But

$$\Delta x = \frac{1-0}{5} = \frac{1}{5}.$$

So

$$\bar{x}_1 = \frac{0+\frac{1}{5}}{2} = \frac{1}{10} \Rightarrow f\left(\tfrac{1}{10}\right) = \sqrt{\left(\tfrac{1}{10}\right)^3+1} = \sqrt{\frac{1001}{1000}}$$

$$\bar{x}_2 = \frac{1}{10}+\frac{1}{5} = \frac{3}{10} \Rightarrow f\left(\tfrac{3}{10}\right) = \sqrt{\left(\tfrac{3}{10}\right)^3+1} = \sqrt{\frac{1027}{1000}}$$

$$\bar{x}_3 = \frac{3}{10}+\frac{1}{5} = \frac{5}{10} \Rightarrow f\left(\tfrac{5}{10}\right) = \sqrt{\left(\tfrac{5}{10}\right)^3+1} = \sqrt{\frac{1125}{1000}}$$

$$\bar{x}_4 = \frac{5}{10}+\frac{1}{5} = \frac{7}{10} \Rightarrow f\left(\tfrac{7}{10}\right) = \sqrt{\left(\tfrac{7}{10}\right)^3+1} = \sqrt{\frac{1343}{1000}}$$

$$\bar{x}_5 = \frac{7}{10}+\frac{1}{5} = \frac{9}{10} \Rightarrow f\left(\tfrac{9}{10}\right) = \sqrt{\left(\tfrac{9}{10}\right)^3+1} = \sqrt{\frac{1729}{1000}}$$

Thus,

$$\int_0^1 \sqrt{x^3+1}\,dx \approx \sum_{i=1}^{5} f(\bar{x}_i)\Delta x$$

$$= \Delta x\left(f(\bar{x}_1)+f(\bar{x}_2)+f(\bar{x}_3)+f(\bar{x}_4)+f(\bar{x}_5)\right)$$

$$= \frac{1}{5}\left(f\left(\tfrac{1}{10}\right)+f\left(\tfrac{3}{10}\right)+f\left(\tfrac{5}{10}\right)+f\left(\tfrac{7}{10}\right)+f\left(\tfrac{9}{10}\right)\right)$$

$$= \frac{1}{5}\left(f\left(\frac{1}{10}\right) + f\left(\frac{3}{10}\right) + f\left(\frac{5}{10}\right) + f\left(\frac{7}{10}\right) + f\left(\frac{9}{10}\right)\right)$$

$$= \frac{1}{5}\left(\sqrt{\frac{1001}{1000}} + \sqrt{\frac{1027}{1000}} + \sqrt{\frac{1125}{1000}} + \sqrt{\frac{1343}{1000}} + \sqrt{\frac{1729}{1000}}\right)$$

$$= \frac{1}{5}(5.5484)$$

$$= \underline{\underline{1.1097}}$$

(17) $$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{e^{x_i}}{1+x_i}\Delta x, \ [0,1] \iff \int_0^1 \frac{e^x}{1+x}\,dx$$

(21) $$\int_2^5 (4-2x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x \quad \text{where } f(x) = 4-2x.$$

But

$$\Delta x = \frac{5-2}{n} = \frac{3}{n}$$

and

$$x_i = 2 + i\Delta x = 2 + \frac{3i}{n}.$$

Thus,

$$\int_2^5 (4-2x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

$$\lim \sum_{i=1}^{n} f\left(2+\frac{3i}{n}\right)\cdot \frac{3}{n}$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} f\left(2+\frac{3i}{n}\right)\cdot \frac{3}{n}$$

$$= \lim_{n\to\infty} \frac{3}{n}\cdot \sum_{i=1}^{n}\left(4-2\left(2+\frac{3i}{n}\right)\right)$$

$$= \lim_{n\to\infty} \frac{3}{n}\cdot \sum_{i=1}^{n}\left(4-4-\frac{6i}{n}\right)$$

$$= \lim_{n\to\infty} \frac{3}{n}\cdot \sum_{i=1}^{n}\left(-\frac{6i}{n}\right)$$

$$= \lim_{n\to\infty} \frac{3}{n}\cdot \frac{-6}{n} \sum_{i=1}^{n} i$$

$$= \lim_{n\to\infty} \frac{-18}{n^2} \sum_{i=1}^{n} i$$

$$= \lim_{n\to\infty} \frac{-18}{n^2}\left(\frac{n(n+1)}{2}\right)$$

$$= \lim_{n\to\infty} \frac{-9(n^2+n)}{n^2}$$

$$= -9 \lim_{n\to\infty} \frac{n^2+n}{n^2}$$

$$= -9 \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)$$

$$= -9 \left(1 + \lim_{n\to\infty} \frac{1}{n}\right)$$

$$= -9 \left(1 + 0\right)$$

$$= -9.$$

(23) $\int_{-2}^{0} (x^2+x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x$ where $f(x) = (x^2+x)$

But

$\Delta x = \frac{0-(-2)}{n} = \frac{2}{n}$ and $x_i = -2 + i\Delta x = \frac{2i}{n} - 2$

Thus,

$$\int_{-2}^{0} (x^2+x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} f\left(\frac{2i}{n} - 2\right) \cdot \frac{2}{n}$$

$$= \lim_{n\to\infty} \frac{2}{n} \sum_{i=1}^{n} \left(\left(\frac{2i}{n} - 2\right)^2 + \left(\frac{2i}{n} - 2\right)\right)$$

$$= \lim \frac{2}{n} \sum_{i=1}^{n} \left(\frac{4i^2}{n^2} - \frac{8i}{n} + 4 + \frac{2i}{n} - 2\right)$$

$$= \lim_{n\to\infty} \frac{2}{n} \sum_{i=1}^{n} \left(\frac{4i^2}{n^2} - \frac{8i}{n} + 4 + \frac{2i}{n} - 2 \right)$$

$$= \lim_{n\to\infty} \frac{2}{n} \sum_{i=1}^{n} \left(\frac{4i^2}{n^2} - \frac{6i}{n} + 2 \right)$$

$$= \lim_{n\to\infty} \frac{2}{n} \left(\frac{4}{n^2} \sum_{i=1}^{n} i^2 - \frac{6}{n} \sum_{i=1}^{n} i + \sum_{i=1}^{n} 2 \right)$$

$$= \lim_{n\to\infty} \frac{2}{n} \left(\frac{4}{n^2} \left(\frac{n(n+1)(2n+1)}{6} \right) - \frac{6}{n} \left(\frac{n(n+1)}{2} \right) + 2n \right)$$

$$= \lim_{n\to\infty} \left(\frac{4(n+1)(2n+1)}{3n^2} - \frac{2\cdot 3(n+1)}{n} + 2\cdot 2 \right)$$

$$= \lim_{n\to\infty} \left(\frac{4(n+1)(2n+1)}{3n^2} - \frac{6(n+1)}{n} + 4 \right)$$

$$= \lim_{n\to\infty} \left(\frac{n^2 \cdot 4\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)}{n^2 \cdot 3} - \frac{6n\left(1+\frac{1}{n}\right)}{n} + 4 \right)$$

$$= \lim_{n\to\infty} \left(\frac{4}{3} \cdot \left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right) - 6\left(1+\frac{1}{n}\right) + 4 \right)$$

$$= \frac{4}{3} \lim_{n\to\infty} \left(1+\frac{1}{n}\right) \cdot \lim_{n\to\infty} \left(2+\frac{1}{n}\right) - 6 \lim_{n\to\infty} \left(1+\frac{1}{n}\right) + 4$$

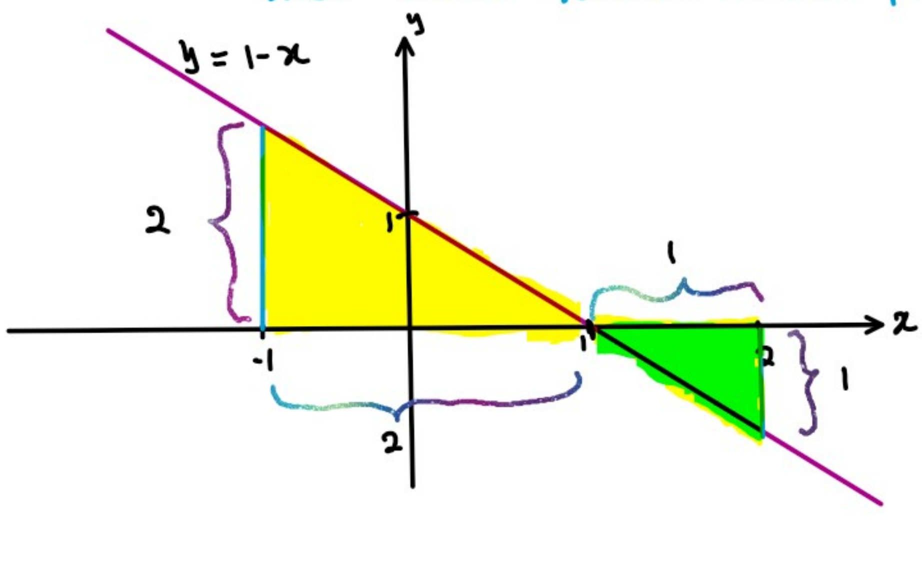
$$= \frac{4}{3}\left((1+0)\cdot(2+0)\right) - 6\left(1+0\right) + 4$$

$$= \frac{4}{3}\cdot 2 - 6 + 4$$

$$= \frac{8}{3} - 2$$

$$= \frac{2}{3}$$

35 $\int_{-1}^{2}(1-x)\,dx \Leftrightarrow$ Find the area bounded by the line $y = 1-x$ and the x-axis on $[-1, 2]$ wherea area above x-axis is interpreted as positive and area below x-axis is interpreted as negative



Thus,

$\int_{-1}^{2}(1-x)\,dx$ = Area of triangle in yellow − Area of triangle in green/lime

$\int_{-1}^{2} (1-x)\,dx$ = Area of triangle in yellow –

$$= \frac{1}{2}(2)(2) - \frac{1}{2}(1)(1)$$

$$= 2 - \frac{1}{2}$$

$$= \frac{3}{2}$$

37) $\int_{-3}^{0} \left(1+\sqrt{9-x^2}\right)dx$

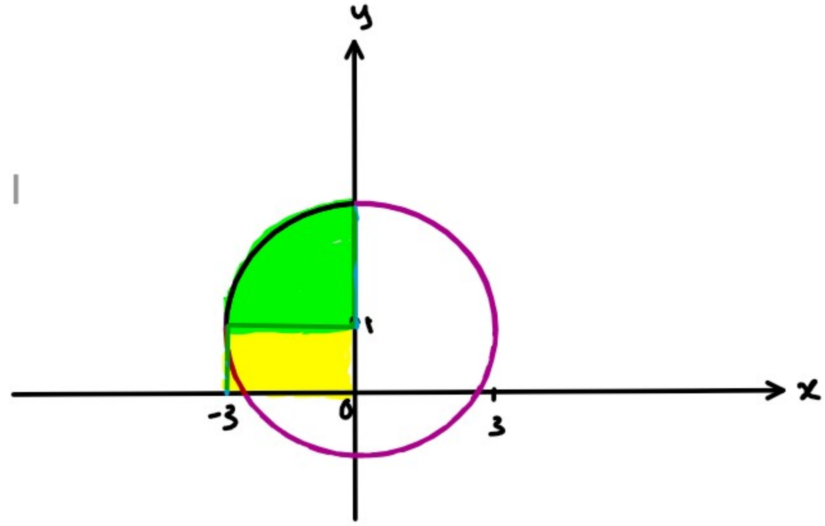
Let $y = 1+\sqrt{9-x^2}$. Then $1 \le y \le 4$ and

$$y-1 = \sqrt{9-x^2} \Rightarrow (y-1)^2 = 9-x^2$$

$$\Rightarrow (y-1)^2 + x^2 = 9$$

$$\Rightarrow x^2 + (y-1)^2 = 3^2$$

So $y = 1+\sqrt{9-x^2}$ is a portion of the circle with radius 3 centered at $(0,1)$



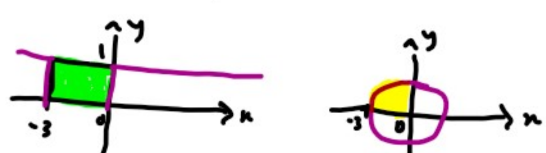
Thus,

$$\int_{-3}^{0} \left(1+\sqrt{9-x^2}\right) dx = \text{Area of sector in green/lime}$$
$$+ \text{Area of rectangle in yellow}$$

$$= \frac{1}{4}(\pi)(3^2) + 1\cdot 3$$

$$= \frac{9\pi}{4} + 3$$

$$= \frac{9\pi + 12}{4}$$



Alternatively:

$$\int_{-3}^{0} \left(1+\sqrt{9-x^2}\right) dx = \int_{-3}^{0} 1\, dx + \int_{-3}^{0} \sqrt{9-x^2}\, dx$$

$$= \text{Area of rectangle} + \text{Area of sector}$$

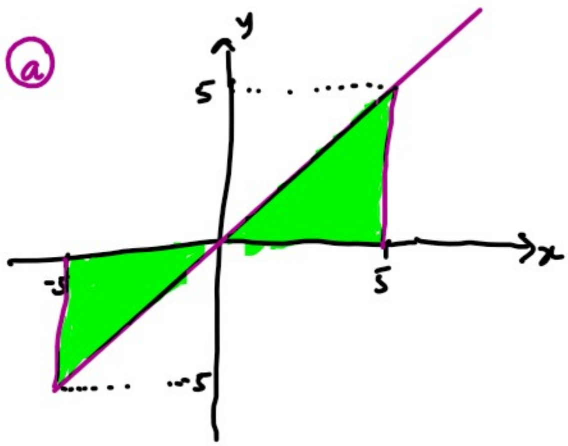
$$= 1 \cdot 3 \quad + \quad \frac{1}{4} \cdot \pi \cdot 3^2$$

$$= 3 + \frac{9\pi}{4}$$

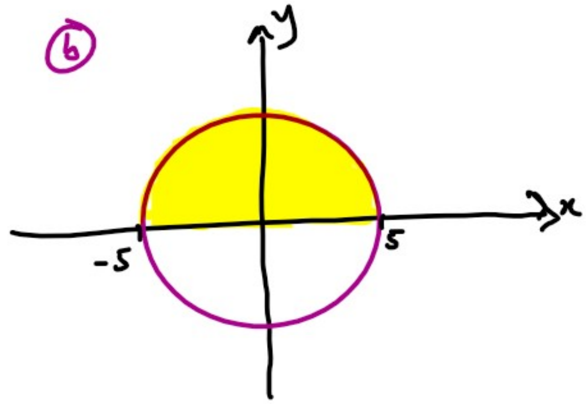
$$= \frac{12 + 9\pi}{4}$$

(38) $\displaystyle\int_{-5}^{5} \left(x - \sqrt{25-x^2}\right) dx = \int_{-5}^{5} x \, dx - \int_{-5}^{5} \sqrt{25-x^2} \, dx$

(a)



(b)



Thus,

$$\int_{-5}^{5} \left(x - \sqrt{25-x^2}\right) dx = \int_{-5}^{0} x \, dx - \int_{-5}^{5} \sqrt{25-x^2} \, dx$$

$$= \left(\begin{array}{l} \text{Area of } \Delta \text{ from } 0 \text{ to } 5 \\ - \text{Area of } \Delta \text{ from } -5 \text{ to } 0 \end{array} \right) - \text{Area of semicircle}$$

$$= \left(\frac{1}{2}(5)(5) - \frac{1}{2}(5)(5) \right) - \frac{1}{2}(\pi)(5^2)$$

$$= \left(\frac{25}{2} - \frac{25}{2}\right) - \frac{25\pi}{2}$$

$$= -\frac{25\pi}{2}$$

NB: Looking at the graph of $y = x$ on $-5 \le x \le 5$ in , we could easily conclude $\int_{-5}^{5} x\,dx = 0$ by symmetry!