**Lecture Note 29** (Ref. text book page 378) 11/08/2021

## 5.2 The Definite Integral

**Definition** If $f$ is a function defined for $a \leq x \leq b$, we divide the interval $[a, b]$ into $n$ subintervals of equal width $\Delta x = (b-a)/n$. We let $x_0 = (a), x_1, x_2, \cdots, x_n (= b)$ be the endpoints of these subintervals and we let $x_1^*, x_2^*, \cdots, x_n^*$ be any **sample points** in these subintervals, so $x_i^*$ lies in the $i$th subinterval $[x_{i-1}, x_i]$. Then the **definite integral of $f$ from $a$ to $b$** is

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

provided that this limit exists and gives the same value for all possible choices of sample points. If it exists, we say that $f$ is **integrable** on $[a, b]$.

**NOTE 1** The symbol $\int$ is called an **integral sign**. In the notation $\int_a^b f(x)dx$, $f(x)$ is called the **integrand** and $a$ and $b$ are called the **limits of integration**; $a$ is the **lower limit** and $b$ is the **upper limit.** For now, the symbol $dx$ has no meaning by itself; $\int_a^b f(x)dx$ is all one symbol. The $dx$ simply indicates that the independent variable is $x$. The procedure of calculating an integral is called **integration**.

**NOTE 2** The definite integral $\int_a^b f(x)dx$ is a number, it does not depend on $x$.

**Example 1** Use the definition of the definite integral to express the limit as a definite integral on the given interval. $\displaystyle\lim_{n\to\infty} \sum_{i=1}^{n} \frac{x_i^*}{(x_i^*)^2 + 4}\Delta x$, $[1, 3]$

**Theorem 1** If $f$ is continuous on $[a, b]$, or if $f$ has only a finite number of jump discontinuities, then $f$ is integrable on $[a, b]$; that is, the definite integral $\int_a^b f(x)dx$ exists.

**Theorem 2** If $f$ is integrable on $[a, b]$, then

$$\int_a^b f(x)dx = \lim_{x\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

where $\Delta x = \dfrac{b-a}{n}$ and $x_i = a + i\Delta x$

## Evaluating Integrals

When we use a limit to evaluate a definite integral, we need to know how to work with sums. So have some sums formula as following.

- $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$
- $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$
- $\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$
- $\sum_{i=1}^{n} c = nc$
- $\sum_{i=1}^{n} ca_i = c\sum_{i=1}^{n} a_i$
- $\sum_{i=1}^{n}(a_i \pm b_i) = \sum_{i=1}^{n} a_i \pm \sum_{i=1}^{n} b_i$

**Example 2** Use the form of the definition of the integral given in Theorem **2** to evaluate the integral. $\int_{-2}^{0}(x^2 + x)dx$

**Example 3** Evaluate the integral by interpreting it in terms of areas. $\int_{-3}^{0}(1 + \sqrt{9 - x^2})dx$

**Properties of the Definite Integral**

1. $\int_a^b f(x)dx = -\int_b^a f(x)dx$
2. $\int_a^a f(x)dx = 0$
3. $\int_a^b c dx = c(b-a)$ where $c$ is any constant
4. $\int_a^b [f(x) \pm g(x)]dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$
5. $\int_a^b cf(x)dx = c\int_a^b f(x)dx$ where $c$ is any constant
6. $\int_a^c f(x) + \int_c^b f(x)dx = \int_a^b f(x)dx$
7. If $f(x) \geq 0$ for $a \leq x \leq b$, then $\int_a^b f(x)dx \geq 0$
8. If $f(x) \geq g(x)$ for $a \leq x \leq b$, then $\int_a^b f(x)dx \geq \int_a^b g(x)dx$
9. If $m \leq f(x) \leq M$ for $a \leq x \leq b$, then $m(b-a) \leq \int_a^b f(x)dx \leq M(b-a)$

**Example 4** Use the properties of integrals to verify the inequality without evaluating the integral.
$2 \leq \int_{-1}^{1} \sqrt{1+x^2}dx \leq 2\sqrt{2}$

**Example 5** Use Property 8 to estimate the value of the integral. $\int_0^3 \frac{1}{x+4}dx$.