Lecture Note 32 (Ref. text book page 412) 11/15/2021

5.5 The Substitution Rule

Because of the Fundamental Theorem, it's important to be able to find antiderivatives. But our antidifferentiation formulas don't tell us how to evaluate integrals such as

$$\int 2x\sqrt{1+x^2}dx$$

So we need the following rule to find antiderivatives of such functions.

The Substitution Rule If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then

$$\int f(g(x))g'(x)dx = \int f(u)du$$

Notice that the Substitution Rule for integration was proved using the Chain Rule for differentiation. Notice also that if $u = g(x)$, then $du = g'(x)dx$, so a way to remember the Substitution Rule is to think of dx and du as differentials.
Thus the Substitution Rule says: **It is permissible to operate with dx and du after integral signs as if they were differentials.**

Example 1 Evaluate the indefinite integral

(a) $\int 2x\sqrt{1+x^2}dx$

(b) $\int x^2e^{x^3}dx$

(c) $\int (2x-4)^8dx$

(d) $\int \sin t\sqrt{1+\cos t}dt$

(e) $\int \cot x dx$

(f) $\int \frac{(\ln x)^2}{x}dx$

(g) $\int \frac{1+x}{1+x^2}dx$

(h) $\int x(2x+5)^8dx$

The Substitution Rule for Definite Integrals If $g^{'}$ is continuous on $[a, b]$ and f is continuous on the range of $u = g(x)$, then

$$\int_a^b f(g(x))g^{'}(x)dx = \int_{g(a)}^{g(b)} f(u)du$$

Example 2 Evaluate

(a) $\displaystyle\int_0^{\pi/6} \frac{\sin t}{\cos^2 t}$

(b) $\displaystyle\int_0^2 (x-1)e^{(x-1)^2}dx$

(c) $\displaystyle\int_0^5 \frac{dx}{5x+1}$

(d) $\displaystyle\int_1^2 \frac{e^{1/x}}{x^2}dx$

Integrals of symmetric functions Suppose f is continuous on$[-a, a]$.

(a) If f is even $[f(-x) = f(x)]$, then $\displaystyle\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$

(b) If f is odd $[f(-x) = -f(x)]$, then $\displaystyle\int_{-a}^{a} f(x)dx = 0$

Example 3 Evaluate

(a) $\displaystyle\int_{-2}^{2} \left(x^6 + 1\right)dx$

(b) $\displaystyle\int_{-1}^{1} \frac{tanx}{1 + x^2 + x^6}dx$