MA 1610 Test 1

September 10, 2021

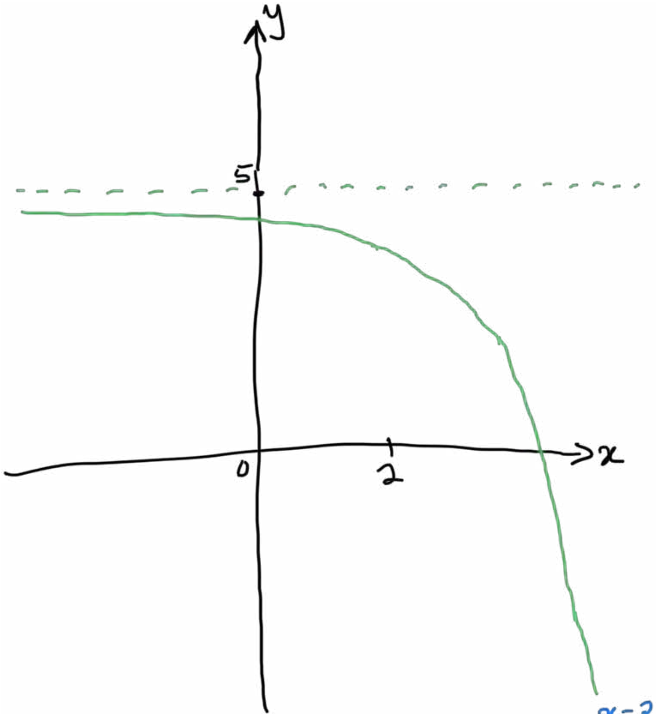
Section: 130

Read all of the following information before starting the exam:

Name: $\underbrace{\text{KEY}}_{\text{By writing my name, I swear by the honor code.}}$

- Show all work, clearly and in order, if you want to get full credit. I reserve the right to take off points if I cannot see how you arrived at your answer (even if your final answer is correct).
- Circle or otherwise indicate your final answers.
- This exam has 8 problems and is worth 100 points. It is your responsibility to make sure that you have all of the seven pages!
- No calculator is allowed. All other electronic devices are prohibited.
- You must give proper reasoning and show your complete work.
- All the BEST!

1. (a) (8 points) Draw a rough sketch of the function $g(x) = -(4^{(x-2)}) + 5$. State the domain, range and the horizontal asymptote of $g(x)$.



$\text{Dom}(g) = (-\infty, \infty)$

$\text{Range}(g) = (-\infty, 5)$

$y = 5$ is a horizontal asymptote since $\lim_{x \to -\infty} g(x) = 5$.

(b) (4 points) Find the domain of the function $h(t) = \dfrac{1}{\sqrt{7-3t}}$.

$$\begin{aligned}\text{Dom}(h) &= \{t \in \mathbb{R} : 7 - 3t > 0\} \\ &= \{t \in \mathbb{R} : 7 > 3t\} \\ &= \{t \in \mathbb{R} : \tfrac{7}{3} > t\} = \boxed{(-\infty, \tfrac{7}{3})}\end{aligned}$$

(c) (1 point) $\ln e =$ $\log_e e = 1$

(d) (3 points) Solve for x: $e^{9-4x} = 8$.

Take ln of both sides.

$\ln e^{9-4x} = \ln 8$

$\Rightarrow (9-4x)\ln e = \ln 8$

$\Rightarrow 9 - 4x = \ln 8$

$\Rightarrow 9 - \ln 8 = 4x$

$\Rightarrow \frac{1}{4}(9 - \ln 8) = x$

So, $\boxed{x = \frac{1}{4}(9 - \ln 8)}$

2. (12 points) Find the equation of the tangent line of the function $f(x) = x^2 + 3x - 4$ at the point $(2, 6)$. (find the slope using the limit definition)

Equation of tangent line at $(2,6)$ is

$$y - y_1 = m(x - x_1)$$

$$y - 6 = m(x-2)$$

$$y - 6 = 7(x-2)$$

$$= 7x - 14$$

$$y = 7x - 14 + 6$$

$$\boxed{y = 7x - 8}$$

$$m = \lim_{h \to 0} \frac{f(2+h) - f(2)}{h}$$

$$= \lim_{h \to 0} \frac{\left((2+h)^2 + 3(2+h) - 4\right) - \left(2^2 + 3(2) - 4\right)}{h}$$

$$= \lim_{h \to 0} \frac{4 + 4h + h^2 + 6 + 3h - 4 - 6}{h}$$

$$= \lim_{h \to 0} \frac{4h + h^2 + 3h}{h}$$

$$= \lim_{h \to 0} \frac{(4 + h + 3)h}{h}$$

$$= \lim_{h \to 0} (4 + h + 3)$$

$$= 4 + 3 = 7$$

3. (10 points) Show that $\lim_{x\to 0} 3x^2 \sin(\frac{2\pi}{x}) = 0$.

$$-1 \le \sin\left(\frac{2\pi}{x}\right) \le 1 \Rightarrow -3x^2 \le 3x^2 \sin\left(\frac{2\pi}{x}\right) \le 3x^2$$

$$\Rightarrow \lim_{x\to 0}\left(-3x^2\right) \le \lim_{x\to 0} 3x^2 \sin\left(\frac{2\pi}{x}\right) \le \lim_{x\to 0} 3x^2$$

$$\Rightarrow \quad 0 \le \lim_{x\to 0} 3x^2 \sin\left(\frac{2\pi}{x}\right) \le 0$$

So, by the Squeeze Theorem,

$$\lim_{x\to 0} 3x^2 \sin\left(\frac{2\pi}{x}\right) = 0$$

4. (12 points) Find the vertical asymptotes of

$$g(x) = \frac{x+1}{x^2+x-12}. = \frac{x+1}{(x+4)(x-3)}$$

$(x+4)(x-3) = 0 \Rightarrow x = -4$ and $x = 3$ are the vertical asymptotes

Since

$$\lim_{x\to -4^-} \frac{x+1}{(x+4)(x-3)} = \frac{-4^-+1}{(-4^-+4)(-4^--3)} = \frac{-3}{(0^-)(-7)} = -\infty$$

$$\lim_{x\to -4^+} \frac{x+1}{(x+4)(x-3)} = \frac{-4^++1}{(-4^++4)(-4^+-3)} = \frac{-3}{(0^+)(-7)} = \infty$$

$$\lim_{x\to 3^-} \frac{x+1}{(x+4)(x-3)} = \frac{3+1}{(3+4)(3^--3)} = \frac{4}{(7)(0^-)} = -\infty$$

$$\lim_{x\to 3^+} \frac{x+1}{(x+4)(x-3)} = \frac{3+1}{(3+4)(3^+-3)} = \frac{4}{(7)(0^+)} = \infty$$

5. Let

$$f(x) = \begin{cases} \sqrt{-x} & \text{if } x < 0, \\ (x-5)^2 & \text{if } 0 \le x < 5, \\ (5-x) & \text{if } x > 5. \end{cases}$$

(a) (12 points) Evaluate the limit, if it exists. Justify your answers.

(i) $\lim_{x\to 0^+} f(x) =$ $\lim_{x\to 0^+} (x-5)^2 = (0-5)^2 = 25$

(ii) $\lim_{x\to 0^-} f(x) =$ $\lim_{x\to 0^-} \sqrt{-x} = \sqrt{-0} = \sqrt{0} = 0$

(iii) $\lim_{x\to 0} f(x) =$ DNE since $\lim_{x\to 0^-} f(x) \neq \lim_{x\to 0^+} f(x)$

(iv) $\lim_{x\to 5^+} f(x) =$ $\lim_{x\to 5^+} (5-x) = 5-5 = 0$

(v) $\lim_{x\to 5^-} f(x) =$ $\lim_{x\to 5^-} (x-5)^2 = (5-5)^2 = 0^2 = 0$

(vi) $\lim_{x\to 5} f(x) =$ $0 = \lim_{x\to 5^-} f(x) = \lim_{x\to 5^+} f(x)$

(b) (4 points) At what points is f discontinuous and why?

At $x = 0$ because $\lim_{x\to 0} f(x) \neq f(0)$ and

At $x = 5$ because $f(5) =$ DNE.

(c) (3points) State the intervals on which f is continuous.

$(-\infty, 0) \cup (0, 5) \cup (5, \infty)$

6. (10 points) Show that there is a solution to $x^3 + 2x^2 - 5x - 6 = 0$ in the interval $(-2, 1)$.

Let $f(x) = x^3 + 2x^2 - 5x - 6 = 0$. Then f is continuous on $[-2, 1]$ as a polynomial and

$f(-2) = (-2)^3 + 2(-2)^2 - 5(-2) - 6 = 4$

$f(1) = (1^3) + 2(1^2) - 5(1) - 6 = -8$

$\Rightarrow \quad f(1) < 0 < f(-2)$

So, by the Intermediate Value Theorem, we can find c in $(-2, 1)$ such that

$$f(c) = 0.$$

7. (9 points) Evaluate the limits, if they exist.

(i) (3 points) $\lim_{x\to-\infty} e^{x^3-x}$ $= \lim_{x\to-\infty} \frac{e^{x^3}}{e^{x}} = \lim_{x\to-\infty} \frac{e^{x^3}(1)}{e^{x^3}(e^{x-x^3})}$

$$= \lim_{x\to-\infty} \frac{1}{e^{x-x^3}} = 0.$$

(ii) (3 points) $\lim_{x\to-4} \frac{x^2 - 16}{x^2 - 3x - 4}$

$$= \frac{(-4)^2 - 16}{(-4)^2 - 3(-4) - 4}$$

$$= \frac{16-16}{16+12-4} = \frac{0}{24} = 0$$

(iii) (3 points) $\lim_{x\to1} e^{x^2-x}$ $= e^{1^2-1} = e^{1-1} = e^0 = 1$

8. (12 points) Find $\lim_{x\to\infty} \sqrt{x^2+3x+5} - x$, if it exists.

$$= \lim_{x\to\infty} \frac{\left(\sqrt{x^2+3x+5} - x\right)\left(\sqrt{x^2+3x+5} + x\right)}{\sqrt{x^2+3x+5} + x}$$

$$= \lim_{x\to\infty} \frac{\left(\sqrt{x^2+3x+5}\right)^2 - x^2}{\sqrt{x^2+3x+5} + x}$$

$$= \lim_{x\to\infty} \frac{x^2+3x+5 - x^2}{\sqrt{x^2+3x+5} + x}$$

$$= \lim_{x\to\infty} \frac{3x+5}{\sqrt{x^2\left(1+\frac{3}{x}+\frac{5}{x^2}\right)} + x}$$

$$= \lim_{x\to\infty} \frac{x\left(3+\frac{5}{x}\right)}{x\sqrt{1+\frac{3}{x}+\frac{5}{x^2}} + x}$$

$$= \lim_{x\to\infty} \frac{x\left(3+\frac{5}{x}\right)}{x\left(\sqrt{1+\frac{3}{x}+\frac{5}{x^2}} + 1\right)}$$

$$= \lim_{x\to\infty} \frac{3+\frac{5}{x}}{\sqrt{1+\frac{3}{x}+\frac{5}{x^2}} + 1} = \frac{3+0}{\sqrt{1+0+0}+1} = \boxed{\frac{3}{2}}$$