Name: KEY **Auburn ID No.:** .... **Section:** 130

1. Let $f(x) = \left(\sqrt{x} + \frac{1}{\sqrt[4]{x}}\right)^2$.

(a) (6 pts) Find the derivative of f

$$f(x) = \left(\sqrt{x} + \frac{1}{\sqrt[4]{x}}\right)^2 = \left(x^{1/2} + x^{-1/4}\right)^2$$

$$\Rightarrow f'(x) = 2\left(x^{1/2} + x^{-1/4}\right)\left(\frac{1}{2}x^{-1/2} - \frac{1}{4}x^{-5/4}\right)$$

$$= \boxed{2\left(\sqrt{x} + \frac{1}{\sqrt[4]{x}}\right)\left(\frac{1}{2\sqrt{x}} - \frac{1}{4\sqrt[4]{x^5}}\right)}$$

(b) (6 pts) Find the equation of tangent line to the graph of f at $(1, 4)$

$$m = f'(x)\Big|_{x=1} = 2\left(\sqrt{1} + \frac{1}{\sqrt[4]{1}}\right)\left(\frac{1}{2\sqrt{1}} - \frac{1}{4\sqrt[4]{1^5}}\right)$$

$$= 2(1+1)\left(\frac{1}{2} - \frac{1}{4}\right) = 1$$

So $y - 4 = m(x-1) \Rightarrow y = 1(x-1)+4$

$\boxed{y = x + 3}$ is the required equation.

2. (a) (6 pts) Differentiate $g(x) = \underbrace{2x^2}_{u}\,\underbrace{\sec x}_{v}$

Using product rule

$$g'(x) = u'v + uv'$$

$$= 4x\sec x + 2x^2\sec x\tan x$$

$$= \boxed{2x\sec x\,(2 + x\tan x)}$$

(b) (6 pts) Evaluate $\displaystyle\lim_{x\to 0}\frac{\sin(9x)}{2x}$

$$\lim_{x\to 0}\frac{\sin(9x)}{2x} = \frac{1}{2}\lim_{x\to 0}\frac{\sin(9x)}{x}$$

$$= \frac{1}{2}\lim_{x\to 0}\frac{\sin(9x)}{x}\cdot\frac{9}{9}$$

$$= \frac{9}{2}\lim_{x\to 0}\frac{\sin(9x)}{9x} = \frac{9}{2}(1) = \boxed{\frac{9}{2}}$$

3. (a) (6 pts) Find the points on the curve $y = 2x^3 + 3x^2 - 12x + 4$ where the tangent line is horizontal. Your answer should be in the form (x, y).

Tangent line is horizontal

$\Rightarrow \frac{dy}{dx} = 6x^2 + 6x - 12 = 0$

$\Rightarrow 6x^2 + 6x - 12 = 0$

$\Rightarrow x^2 + x - 2 = 0$

$\Rightarrow (x+2)(x-1) = 0$

$\Rightarrow x = -2, 1$

So, the points on the curve where the tangent lines are horizontal are $\boxed{(1, y(1))}$ and $\boxed{(-2, y(-2))}$.

(b) (6 pts) Find the derivative of the function $y = \cos^{-1}(4x + 1)$

$y = \cos^{-1}(4x+1)$

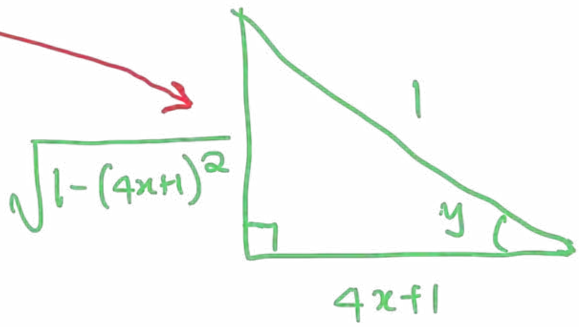
$\Rightarrow \boxed{\cos y = 4x+1}$

$\Rightarrow \frac{d}{dx}(\cos y) = \frac{d}{dx}(4x+1)$

$\Rightarrow -\sin y \cdot \frac{dy}{dx} = 4$

$\Rightarrow \frac{dy}{dx} = \frac{-4}{\sin y}$

$= \boxed{\frac{-4}{\sqrt{-16x^2 - 8x}}}$



(c) (7 pts) Use logarithmic differentiation to find the derivative of the function $y = (\sqrt{x})^{8\cos x}$

$y = (\sqrt{x})^{8\cos x} \Rightarrow \ln y = \ln(\sqrt{x})^{8\cos x} = (8\cos x)(\ln\sqrt{x})$

$= (8\cos x)(\ln x^{1/2}) = (8\cos x)\frac{1}{2}\ln x$

$= 4\cos x \ln x$

So $\frac{d}{dx}(\ln y) = \frac{d}{dx}(4\cos x \ln x)$

$\frac{y'}{y} = 4\cos x(\ln x)' + (4\cos x)'\ln x$

$= 4\cos x\left(\frac{1}{x}\right) + (-4\sin x)\ln x$

i.e., $y' = y\left[\frac{4\cos x}{x} - 4\sin x \ln x\right]$

$= \boxed{(\sqrt{x})^{8\cos x}\left[\frac{4\cos x}{x} - 4\sin x \ln x\right]}$

4. A particle moves according to a law of motion $s(t) = \frac{t^3}{3} - \frac{t^2}{2} - 2t + 24$ where t is measured in seconds and s in meters.

(a) (5 pts) Find the velocity at time t

$$v(t) = s'(t)$$
$$= \boxed{t^2 - t - 2}$$

(b) (5 pts) What is the velocity after 1 second

$$v(1) = 1^2 - 1 - 2$$
$$= \boxed{-2 \text{ m/s}}$$

(c) (5 pts) When is the particle at rest?

At rest when $v(t) = 0$.

i.e., $t^2 - t - 2 = 0$

$\Rightarrow (t-2)(t+1) = 0$

$\Rightarrow t = -1, 2$

$\Rightarrow \boxed{t = 2}$ seconds since $t > 0$

(d) (6 pts) When is the particle moving in the positive direction? (Enter your answer using interval notation.)

Solve $v(t) > 0$ for t.

Looking from the sign table, the particle is moving in the positive direction for t in $\boxed{(2, \infty)}$ since time is positive.

		-1	2
$t+1$	$-$	$+$	$+$
$t-2$	$-$	$-$	$+$
$(t-2)(t+1)$	$+$	$-$	$+$

(e) (5 pts) Find the acceleration after 1 second

$a(t) = v'(t) = 2t - 1$

$\Rightarrow a(1) = 2(1) - 1$

$= \boxed{1\, m/s^2}$

5. A roast turkey is taken from an oven when its temperature has reached 75°C and is placed on a table in a room where the temperature is 25°C.

(a) (6 pts) If the temperature of the turkey is 50°C after half an hour, find an expression for the temperature T as a function of time t.

$T(t) = T_s + (T(0) - T_s)e^{kt}$

$= 25 + (75 - 25)e^{kt}$ where $T_s = 25°C$, $T(0) = 75°C$.

$= 25 + 50e^{kt}$

Thus, $T(30) = 50$

$\Rightarrow 25 + 50e^{30k} = 50 \Rightarrow 50e^{30k} = 50 - 25 = 25$

$\Rightarrow e^{30k} = \frac{25}{50} = \frac{1}{2} \Rightarrow \ln e^{30k} = \ln(1/2)$

$\Rightarrow 30k = \ln(1/2) \Rightarrow k = \frac{1}{30}\ln(1/2)$

Hence,

$T(t) = 25 + 50 e^{(\frac{1}{30}\ln 1/2)t} = \boxed{25 + 50\left(2^{-t/30}\right)}$

(b) (6 pts) What is the temperature after 60 minutes?

$T(60) = 25 + 50\left(2^{-60/30}\right)$

$= 25 + 50\left(2^{-2}\right)$

$= 25 + \frac{25}{2}$

$= \boxed{37.5°C}$

(c) (6 pts) When will the turkey have cooled to 35°C?

$T(t) = 35, \quad t = ?$

$$35 = 25 + 50 e^{\left(\frac{1}{30}\ln\frac{1}{2}\right)t}$$

$$\Rightarrow \frac{35-25}{50} = e^{\left(\frac{1}{30}\ln\frac{1}{2}\right)t}$$

$$\Rightarrow \ln(1/5) = \ln e^{\left(\frac{1}{30}\ln\frac{1}{2}\right)t} = \left(\frac{1}{30}\ln\frac{1}{2}\right)t$$

$$\Rightarrow t = \frac{30\ln 1/5}{\ln 1/2} = \boxed{\frac{30\ln 5}{\ln 2}}$$

6. Differentiate the functions

(a) (7 pts) $g(x) = 3x^2 e^{(3-5x^3)}$

Using product rule:

$$g'(x) = (3x^2)\left(e^{3-5x^3}\right)' + (3x^2)'\left(e^{3-5x^3}\right)$$

$$= (3x^2)\left(-15x^2 e^{3-5x^3}\right) + 6x e^{3-5x^3}$$

$$= 3x e^{3-5x^3}\left[-15x^3 + 2\right]$$

$$= \boxed{3x(2-15x^3)e^{3-5x^3}}$$

(b) (6 pts) $h(x) = \sqrt[3]{\frac{x-1}{x^5+1}}$

Using logarithmic differentiation,

$$y = \sqrt[3]{\frac{x-1}{x^5+1}}$$

$$\Rightarrow \ln y = \ln\left(\sqrt[3]{\frac{x-1}{x^5+1}}\right)$$

$$= \ln\left(\frac{x-1}{x^5+1}\right)^{1/3}$$

$$= \frac{1}{3}\ln\left(\frac{x-1}{x^5+1}\right)$$

$$= \frac{1}{3}\ln(x-1) - \frac{1}{3}\ln(x^5+1)$$

$$\Rightarrow \frac{d}{dx}(\ln y) = \frac{d}{dx}\left(\frac{1}{3}\ln(x-1) - \frac{1}{3}\ln(x^5+1)\right)$$

$$\Rightarrow \frac{y'}{y} = \frac{1}{3}\left(\frac{1}{x-1}\right) - \frac{1}{3}\left(\frac{5x^4}{x^5+1}\right)$$

$$\Rightarrow y' = y\left[\frac{1}{3(x-1)} - \frac{5x^4}{3(x^5+1)}\right]$$

$$= \boxed{\sqrt[3]{\frac{x-1}{x^5+1}}\left[\frac{1}{3(x-1)} - \frac{5x^4}{3(x^5+1)}\right]}$$