Name: KEY **Auburn ID No.:** .... **Section:** 130

1. (a) i. (5 pts) Find the linear approximation, $L(x)$ of the function $f(x) = \sqrt{x}$ at $x = 4$.

$$L(x) = f(4) + f'(4)(x-4)$$
$$= 2 + \frac{1}{4}(x-4)$$
$$= \frac{1}{4}x + 2 - 1$$
$$= \boxed{\frac{1}{4}x + 1}$$

$f(x) = \sqrt{x} = x^{1/2}$

$\Rightarrow f'(x) = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$

So,

$f'(4) = \frac{1}{2\sqrt{4}} = \frac{1}{2(2)} = \frac{1}{4}$

ii. (5 pts) Use $L(x)$ to approximate the number $\sqrt{3.9}$

Since $3.9 \approx 4$,

$$\sqrt{3.9} \approx L(3.9) = \frac{1}{4}(3.9) + 1$$
$$= 0.975 + 1$$
$$= \boxed{1.975}$$

(b) (7 pts) Find $\lim_{x\to\infty} x \sin(\frac{-9\pi}{x})$. Use l'Hospital's Rule if appropriate. If there is a more elementary method, consider using it.

$$\lim_{x\to\infty} x \sin\left(\frac{-9\pi}{x}\right)$$
$$= \lim_{x\to\infty} \frac{\sin\left(-\frac{9\pi}{x}\right)}{1/x}$$
$$= \lim_{y\to 0} \frac{\sin(-9\pi y)}{y}$$
$$= \lim_{y\to 0} \frac{\sin(-9\pi y)}{-9\pi y} \cdot (-9\pi)$$
$$= -9\pi \lim_{y\to 0} \frac{\sin(-9\pi y)}{-9\pi y} = \boxed{-9\pi}$$

Alternatively, using l'Hospital's rule,

$$\lim_{x\to\infty} x \sin\left(\frac{-9\pi}{x}\right) = \lim_{x\to\infty} \frac{\sin\left(\frac{-9\pi}{x}\right)}{1/x} \sim \frac{0}{0}$$
$$\overset{L'H}{=} \lim_{x\to\infty} \frac{\frac{9\pi}{x^2}\cos\left(\frac{-9\pi}{x}\right)}{-\frac{1}{x^2}}$$
$$= -9\pi \lim_{x\to\infty} \cos\left(\frac{-9\pi}{x}\right)$$
$$= -9\pi \cos(0)$$
$$= \boxed{-9\pi}$$

2. (a) (7 pts) Find $\lim_{x\to 1} \frac{1-x^2}{\ln x}$. Use l'Hospital's Rule where appropriate. If there is a more elementary method, consider using it.

$$\lim_{x\to 1} \frac{1-x^2}{\ln x} = \frac{0}{0}$$

So,

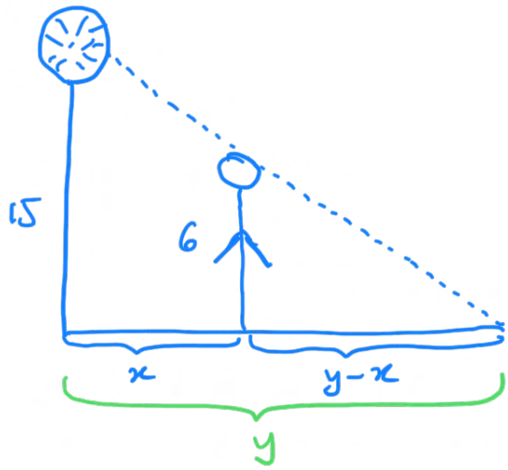
$$\lim_{x\to 1} \frac{1-x^2}{\ln x} \overset{L'H}{=} \lim_{x\to 1} \frac{-2x}{1/x}$$

$$= \frac{-2(1)}{1/1}$$

$$= \boxed{-2}$$

(b) (7 pts) A street light is mounted at the top of a 15-ft-tall pole. A man 6 ft tall walks away from the pole with a speed of 5 ft/s along a straight path. How fast is the tip of his shadow moving when he is 45 ft from the pole?

$$\frac{dx}{dt} = 5 \qquad x = 45$$



By similar triangles,

$$\frac{y-x}{y} = \frac{6}{15}$$

$$15(y-x) = 6y$$

$$15y - 6y = 15x$$

$$9y = 15x$$

$$y = \frac{5}{3}x$$

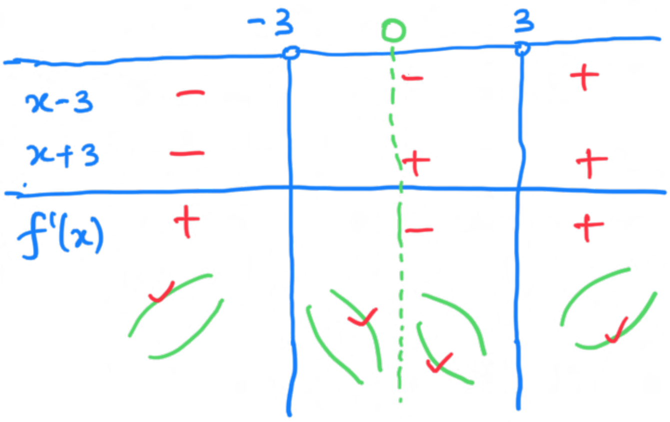
$$\Rightarrow \frac{dy}{dt} = \frac{5}{3}\frac{dx}{dt}$$

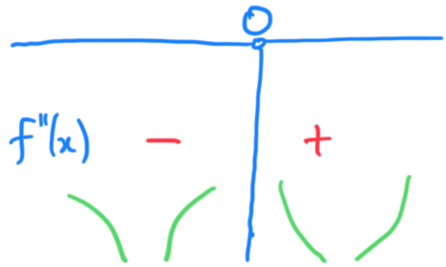
$$= \frac{5}{3}(5)$$

$$= \boxed{\frac{25}{3} \text{ ft/s}}$$

So, the tip of his shadow is moving at a speed independent of his distance from the pole.

3. Let $f(x) = x^3 - 27x + 5$. $\Rightarrow f'(x) = 3x^2 - 27 = 3(x-3)(x+3)$, $f''(x) = 6x$





(a) (3 pts) Find the interval(s) where the function f is increasing

$(-\infty, -3)$, $(3, \infty)$

(b) (3 pts) Find the interval(s) where f is decreasing

$(-3, 3)$

(c) (3 pts) Find the interval(s) where the graph of f is concave upward

$(0, \infty)$

(d) (3 pts) Find the interval(s) where the graph of f is concave downward

$(-\infty, 0)$

(e) (3 pts) Find the local extrema points of f

Local max: $(-3, f(-3))$; Local min: $(3, f(3))$

(f) (3 pts) Find the inflection points, if any, of f

$(0, f(0))$

(g) (3 pts) Find $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$

$$\lim_{x\to\infty} \underbrace{(x^3-27x+5)}_{1} = \lim_{x\to\infty} \frac{x^3\left(1-\frac{27}{x^2}+\frac{5}{x^3}\right)}{x^3\left(\frac{1}{x^3}\right)} = \lim_{x\to\infty} \frac{1-\frac{27}{x^2}+\frac{5}{x^3}}{\frac{1}{x^3}} = \frac{1}{0} = \infty.$$

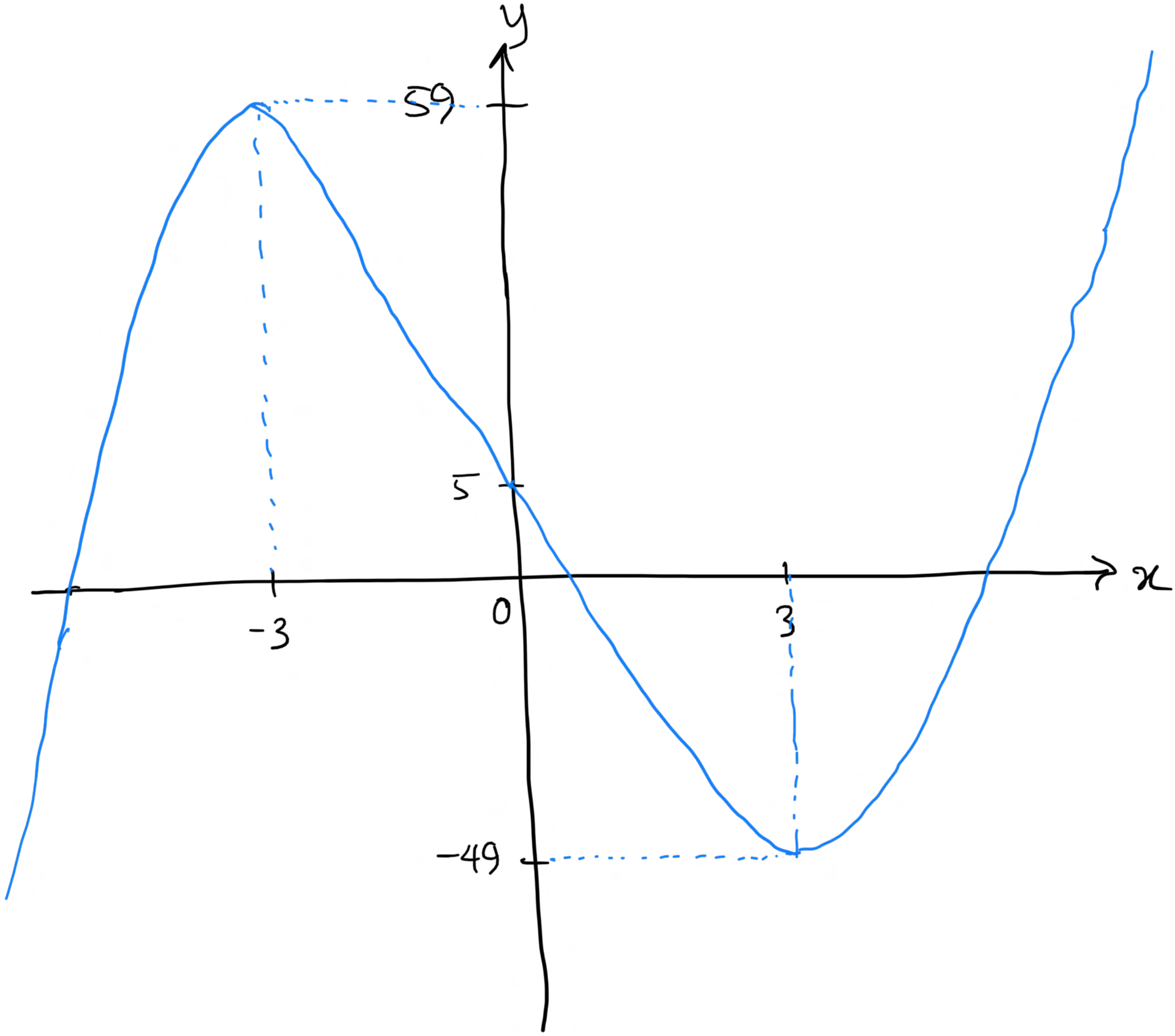
$$\lim_{x\to-\infty} \underbrace{(x^3-27x+5)}_{1} = \lim_{x\to-\infty} \frac{x^3\left(1-\frac{27}{x^2}+\frac{5}{x^3}\right)}{x^3\left(\frac{1}{x^3}\right)} = \lim_{x\to-\infty} \frac{1-\frac{27}{x^2}+\frac{5}{x^3}}{\frac{1}{x^3}} = \frac{1}{0^-} = -\infty$$

(h) (6 pts) Evaluate f at some convenient points, and together with your results in (a) - (f), sketch the graph of f

$f(-3) = (-3)^3 - 27(-3) + 5 = 59$

$f(0) = 0^3 - 27(0) + 5 = 5$

$f(3) = 3^3 - 27(3) + 5 = -49$.



4. (a) (5 pts) Does the function $f(x) = 5x^2 - 4x + 1,\ [0, 2]$ satisfy the hypotheses of the Mean Value Theorem on the given interval?
   i. Yes, it does not matter if f is continuous or differentiable, every function satisfies the Mean Value Theorem.
   ii. Yes, f is continuous on [0, 2] and differentiable on (0, 2) since polynomials are continuous and differentiable on $\mathbb{R}$.
   iii. No, f is not continuous on [0, 2].
   iv. No, f is continuous on [0, 2] but not differentiable on (0, 2).
   v. There is not enough information to verify if this function satisfies the Mean Value Theorem.

(b) (5 pts) If it satisfies the hypotheses, find all numbers c that satisfy the conclusion of the Mean Value Theorem. If it does not satisfy the hypotheses, write DNE).

$$f'(c) = \frac{f(2) - f(0)}{2 - 0} = \frac{13 - 1}{2} = 6$$

$$\Rightarrow 10c - 4 = 6$$

$$\Rightarrow c = \frac{10}{10} = \boxed{1}$$

(c) (5 pts) Find the intervals of concavity of the function $f(x) = \frac{x}{1+x^2}$

Using quotient rule,

$$f'(x) = \frac{(1+x^2)1 - x(2x)}{(1+x^2)^2} = \frac{1-x^2}{(1+x^2)^2}$$

$$f''(x) = \frac{(1+x^2)^2(-2x) - (1-x^2)\,2(1+x^2)(2x)}{(1+x^2)^4}$$

$$= \frac{(1+x^2)\left[-2x - 2x^3 - 4x + 4x^3\right]}{(1+x^2)^4}$$

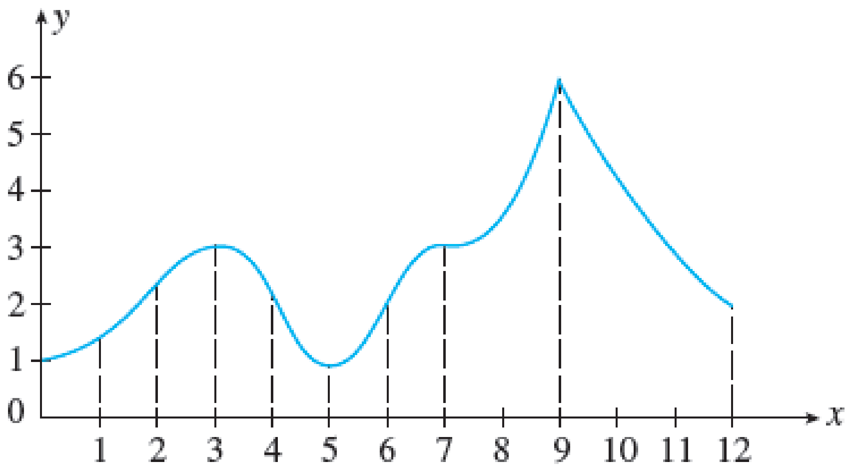
$$= \frac{(1+x^2)(2x^3 - 6x)}{(1+x^2)^4} = \frac{2x(x-\sqrt{3})(x+\sqrt{3})}{(1+x^2)^3}$$

	$(-\infty, -\sqrt{3})$	$(-\sqrt{3}, 0)$	$(0, \sqrt{3})$	$(\sqrt{3}, \infty)$
x	−	−	+	+
$x-\sqrt{3}$	−	−	−	+
$x+\sqrt{3}$	−	+	+	+
$f''(x)$	−	+	−	+

Concave down on $(-\infty, -\sqrt{3}) \cup (0, \sqrt{3})$

Concave up on $(-\sqrt{3}, 0) \cup (\sqrt{3}, \infty)$

5. Consider the graph of a function, f below



(a) (3 pts) Find the interval(s) where f is concave upward

$(0,2), (4,6), (7,12)$

(b) (3 pts) Find the interval(s) where f is concave downward

$(2,4), (6,7)$

(c) (3 pts) Find the interval(s) where f is increasing

$(0,3), (5,9)$

(d) (3 pts) Find the interval(s) where f is decreasing

$(3,5), (9,12)$

(e) (3 pts) Find the inflection point(s) of f

$(2,f(2)), (4,f(4)), (6,f(6)), (7,f(7))$

(f) (3 pts) Find the local minimum point(s) of f

$(5,f(5))$

(g) (3 pts) Find the local maximum point(s) of f

$(3, f(3))$, $(9, f(9))$

(h) (3 pts) Find the absolute maximum point of f

$(9, f(9))$

(i) (3 pts) Find the absolute minimum value of f

$f(0) = 1 = f(5)$

Scratch Work