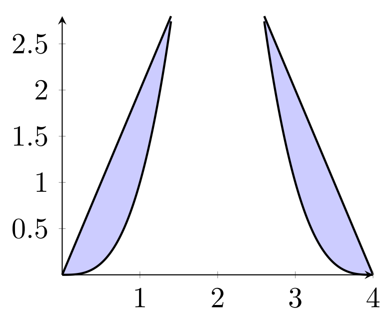
1620 3/5 Recitation Key

1. Find the surface area of the solid made by rotating the curve $y = \frac{\sqrt{3}}{4}\sqrt{16-4x^2}$ from $-2 \leq x \leq 2$ about the $x-$axis.

   $y' = -\frac{\sqrt{3}x}{\sqrt{16-4x^2}}$ so $1 + y'^2 = \frac{16-x^2}{16-4x^2}$. Since we are revolving about the $x-$ axis, our radius is the y value of our curve, so the surface area is

$$\begin{aligned}
\int_{-2}^{2} 2\pi\frac{\sqrt{3}}{4}\sqrt{16-4x^2}\sqrt{\frac{16-x^2}{16-4x^2}}\,dx &= \frac{\sqrt{3}}{2}\pi\int\sqrt{16-x^2}\,dx \\
&= \frac{\sqrt{3}}{2}\pi\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} 4\cos u\sqrt{16-16\sin^2 u}\,du \quad x = 4\sin u \\
&= 8\sqrt{3}\pi\int\cos^2 u\,du \quad \text{Pythagorean} \\
&= 4\sqrt{3}\pi\int\cos(2u)+1\,dy \quad \text{HA} \\
&= 4\sqrt{3}\pi(\frac{1}{2}\sin(2u)+u)\Big|_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \\
&= 6\pi + \frac{4}{\sqrt{3}}\pi^2
\end{aligned}$$

2. Consider the region in the first quadrant bounded by $y = 2x$ and $y = x^3$. Set up an integral for the volume of the solid generated by revolving this region about $x = 2$ (a) using the disk method (b) using the shell method.



   Note that $2x = x^3$ when $x = 0, \pm\sqrt{2}$, and since are we in the first quadrant, our intersection points are $(0,0)$ and $(\sqrt{2}, 2\sqrt{2})$. For the disk method, we use washers in terms of y so $r_O = 2 - \frac{1}{2}y$ and $r_I = 2 - \sqrt[3]{y}$. Thus, the volume is $\int_0^{2\sqrt{2}} \pi((2-\frac{1}{2}y)^2 - (2-\sqrt[3]{y})^2)\,dy$. For the shell method, $r = 2 - x$ and $h = 2x - x^3$ so the volume is $\int_0^{2} 2\pi(2-x)(2x-x^3)\,dx$.

3. Find the length of the curve $y = (\frac{x}{2})^{\frac{3}{2}}$ on $0 \le x \le 32$. Compare this arc length to the length of the line segment from $(0, f(0))$ to $(32, f(32))$. Which is longer, and does this answer make sense? (b) What is the average value of f over this interval? Must $f(x)$ necessarily equal its average value for some $0 \le x \le 32$? Why or why not?

   $f'(x) = \frac{3}{4}(\frac{x}{2})^{\frac{1}{2}}$ so the arc length is $\int_0^{32} \sqrt{1 + \frac{9x}{32}}\ dx = \frac{64}{27}(10\sqrt{10} - 1) \approx 72$. The length of the line segment is $\sqrt{32^2 + 16^3} = 32\sqrt{5} \approx 71$, which makes sense since a line is the shortest distance between two points.

   $f_{ave} = \frac{1}{32}\int_0^3 2(\frac{x}{2})^{\frac{3}{2}}\ dx = \frac{128}{5}$. Since f is continuous everywhere, the MVT for integrals ensures that there is some value $c \in (0, 32)$ where $f(c) = f_{ave}$.

4. Set up an integral to find the surface area of the solid generated by revolving the curve $y = \sqrt{9 - x^2} - 4$ about the $x-$axis.

   $y' = -\frac{x}{\sqrt{9-x^2}}$ so $1 + y' = \frac{9}{9-x^2}$. Since we are rotating about $y = 0$, our radius is the y value of the curve. Thus, the surface area is $\int_{-3}^{3} 2\pi((\sqrt{9 - x^2} - 4)(\frac{9}{9-x^2})\ dx$.

5. Consider the region bounded by $y = \sin x \cos x$ and $y = 0$ from $0 \le x \le \frac{\pi}{2}$. Set up an integral for the volume of the solid generated by rotating about (a) the $x-$axis (b) $y = 2$ (c) the $y-$axis (d) the surface area of the solid from part (a).

   (a) Disk with $r = \sin x \cos x$ so $V = \int_0^{\frac{\pi}{2}} \pi \sin^2 x \cos^2 x\ dx$.

   (b) Washer wit $r_O = 2$ and $r_I = 2 - \sin x \cos x$ so $V = \int_0^{\frac{\pi}{2}} \pi(4 - (2 - \sin x \cos x)^2\ dx$.

   (c) Shell with $r = x$ and $h = \sin x \cos x$ so $V = \int_0^{\frac{\pi}{2}} 2\pi x \sin x \cos x\ dx$..

   (d) $y' = \cos^2 x - \sin^2 x$ so $1 + y'^2 = 1 + \cos^4 x + \sin^4 x - 2\sin^2 x \cos^2 x$. Thus, the surface area is $A = 2\pi \int_0^{\frac{\pi}{2}} \sin x \cos x \sqrt{1 + \cos^4 x + \sin^4 x - 2\sin^2 x \cos^2 x}\ dx$.

6. Let $f(x) = \frac{e^x+e^{-x}}{2}$. (a) Find the exact length of the curve $y = f(x)$ on $-1 \le x \le 1$. (b) The surface area of the solid made by rotating the curve about the $y-$axis.

$f'(x) = \frac{e^x-e^{-x}}{2}$ so the arc length is $\int_{-1}^{1} \sqrt{1 + \frac{e^{2x}+e^{-2x}}{4} - \frac{1}{2}}\, dx$.

$$\begin{aligned}
\int_{-1}^{1} \sqrt{1 + \frac{e^{2x}+e^{-2x}}{4} - \frac{1}{2}}\, dx &= \frac{1}{2}\int_{-1}^{1} \sqrt{e^{2x}+e^{-2x}+2}\, dx \\
&= -\frac{1}{4}\sqrt{e^{u}+e^{-u}+2}\, du && u = -2x \\
&= -\frac{1}{4}\int e^{-\frac{u}{2}}(e^{u}+1)\, du \\
&= \frac{1}{2}\int (e^{-2v}+1)e^{v}\, dx && v = -\frac{u}{2} \\
&= \frac{1}{2}\int e^{v}+e^{-v}\, dv \\
&= \frac{e^x-e^{-x}}{2}\Big|_{-1}^{1} \\
&= e - \frac{1}{e}
\end{aligned}$$

The surface area of a solid rotated about the $x-$axis is given by $2\pi \int y\, ds$ where ds is the expression we integrate for arc length. $\int_{-1}^{1} \pi(e^x+e^{-x})\sqrt{\frac{1}{2} + \frac{e^{2x}+e^{-2x}}{4}}\, dx$.

7. Find the total area bounded between $y = \sin x$ and $y = \cos x$ on $-\pi \le x \le \pi$.

$\sin x = \cos x$ at $x = -\frac{3\pi}{4}, \frac{\pi}{4}$ in our given interval. Splitting our area into 3 regions gives us that the total area is equal to $\int_{-\pi}^{-\frac{3\pi}{4}} \sin x - \cos x\, dx + \int_{-\frac{3\pi}{4}}^{\frac{\pi}{4}} \cos x - \sin x\, dx + \int_{\frac{\pi}{4}}^{\pi} \sin x - \cos x\, dx = 4\sqrt{2}$.

8. Let $f(x) = \int_{1}^{x} \sqrt{t^4 - 1}\, dt$ for $x \ge 1$.

9. Find the exact length of the curve $y = f(x)$ over $1 \le x \le 3$. What is the surface area generated by rotating the curve about the $y-$axis?

By the FTC, $f'(x) = \sqrt{x^4 - 1}$ so the arc length is $\int_{1}^{3} \sqrt{x^4}\, dx = \frac{1}{3}x^3\Big|_{1}^{3} = \frac{26}{3}$.

$2\pi \int_{1}^{3} x^3\, dx = \frac{\pi}{2}x^4\Big| 1^3 = 40\pi$.

10. Consider the region bounded by $y = \sqrt{x}$ and $y = 1$. (a) What is the area of this region? (b) What is the volume of the solid generated by rotating this region about the line $x = -12$?

(a) The area is $\int_{0}^{1} 1 - \sqrt{x}\, dx = \frac{1}{3}$.

(b) Using shells, the volume is $2\pi \int_{0}^{1} (12 + x)(1 - \sqrt{x})\, dx = \frac{10}{3}$.

11. Find the arc length function $s(x)$ for the curve $y = \frac{(x^2+2)^{\frac{3}{2}}}{3}$ with starting point $(0, \frac{2\sqrt{2}}{3})$.

$\frac{dy}{dx} = x(x^2 + 2)^{\frac{1}{2}}$. $s(x) = \int_{0}^{x} \sqrt{1 + x^4 + 2x^2}\, dx$.

12. Consider the region bounded by $y = x - x^3$, $y = x^2 - x$, and $x \geq 0$. (a) What is the area of the region? (b) Set up an integral for the volume of the solid generated by rotating the region about $y = 3$. (c) Repeat (b) with the $y-$axis.

(a) $x - x^3 = x^2 - x$ at $x = -2, 0, 1$ so our intersection points are $(0, 0)$ and $(1, 0)$. In the interval $0 \leq x \leq 1$, $x - x^3 > x^2 - x$ so the area of the region is $\int_0^1 x - x^3 + x - x^2 \, dx = \frac{5}{12}$.

(b) Using washers, $r_O = 3 - x^2 + x$ and $r_I = 3 - x + x^3$ so the volume is $\int_0^1 \pi((3 - x^2 + x)^2 - (3 - x + x^3)^2) \, dx$.

(c) Using shells, $r = x$ and $h = -x^3 - x^2 + 2x$ so the volume is $\int_0^1 2\pi x(-x^3 - x^2 + 2x) \, dx$.

13. Find the length of the curve $y = \ln \sec x$ from $0 \leq x \leq \frac{\pi}{4}$.

$y' = \tan x$ so $1 + y'^2 = 1 + \tan^2 x = \sec^2 x$. Thus, $L = \int_0^{\frac{\pi}{4}} \sqrt{1 + \tan^2 x} \, dx = \ln|\sec x + \tan x|\Big|_0^{\frac{\pi}{4}} = \ln(\sqrt{2} + 1)$.

14. Find the length of the curve $y = e^{2x}$ from $0 \leq x \leq 1$.

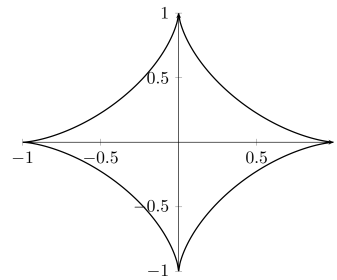
$y' = 2e^{2x}$ so $1 + y'^2 = 1 + (2e^{2x})$. Then

$$
\begin{aligned}
\int_0^1 \sqrt{1 + (2e^{2x})^2} \, dx &= \frac{1}{2} \int \sqrt{1 + \tan^2\theta} \sec^2\theta \cot\theta \, d\theta && \tan\theta = 2e^{2x} \\
&= \frac{1}{2} \int \sec^3\theta \cot\theta \, d\theta && \text{Pythagorean} \\
&= \frac{1}{2} \int \sec\theta \tan\theta + \csc\theta \, d\theta && \text{Pythagorean} \\
&= \frac{1}{2}(\sec\theta - \ln|\csc\theta + \cot\theta|) \\
&= \frac{1}{2}((1 + (2e^{2x})^2) - \ln|\frac{1 + (2e^{2x})^2}{2e^{2x}} + \frac{1}{2e^{2x}}|)\Big|_0^1 \\
&= \frac{1}{2}(-4 + 4e^4 + \ln 3 - \ln(\frac{2e^4 + 1}{e^2}))
\end{aligned}
$$

15. A length of string, chain, etc., which is hanging by both ends is called a catenary. Mathematically, a catenary is described by $f(x) = a\frac{e^{\frac{x}{a}} + e^{-\frac{x}{a}}}{2}$ with $a > 0$. What is the length of a catenary with its ends fixed at $(1, 1)$ and $(-1, 1)$?

$f'(x) = \frac{e^{\frac{x}{a}} - e^{-\frac{x}{a}}}{2}$ and $1 + f'(x)^2 = \frac{e^{\frac{2x}{a}} + e^{-\frac{2x}{a}}}{4} + \frac{1}{2} = (\frac{e^{\frac{x}{a}}}{2} + \frac{e^{-\frac{x}{a}}}{2})^2$. Thus, the arc length is $\int_{-1}^1 \sqrt{(\frac{e^{\frac{x}{a}}}{2} + \frac{e^{-\frac{x}{a}}}{2})^2} \, dx = \int_{-1}^1 \frac{e^{\frac{x}{a}}}{2} + \frac{e^{-\frac{x}{a}}}{2} \, dx = \frac{a}{2}(e^{\frac{x}{a}} - e^{-\frac{x}{a}})\Big|_{-1}^1 = ae^{-\frac{1}{a}}(e^{\frac{2}{a}} - 1)$.

16. Sketch the asteroid given by $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 1$ and find the perimeter of the asteroid.



Since the shape is symmetric with respect to the origin, we can take the arc length of a piece of the curve in one quadrant, and quadruple this for the total arc length. The piece of the curve in quadrant 1 is given by $f(x) = (1 - x^{\frac{2}{3}})^{\frac{3}{2}}$ so $f'(x) = -\sqrt{\frac{1-x^{\frac{2}{3}}}{x^{\frac{1}{3}}}}$ and $1 + f'(x)^2 = \frac{1}{x^{\frac{2}{3}}}$. Thus, the arc length of this curve is $\int_0^1 \frac{1}{x^{\frac{1}{3}}}\, dx = \frac{3}{2}$ so the perimeter is 6.

17. Let $f(x) = \cos^{-1}(e^x)$ on $(-\infty, 0]$. Set up 2 integrals, one with respect to x and the other y, to find the arc length of $y = f(x)$ on $-\frac{\ln 2}{2} \leq x \leq 0$, and then compute the integral of your choice.

$f'(x) = -\frac{e^x}{\sqrt{1-e^{2x}}}$ so $1 + f'(x)^2 = \frac{1}{1-e^{2x}}$. Thus, the arc length is

$$\begin{aligned} \int_{-\frac{\ln 2}{2}}^{0} \sqrt{\frac{1}{1-e^{2x}}}\, dx &= \int_{\frac{pi}{4}}^{\frac{\pi}{2}} \frac{\cot\theta}{\sqrt{1-\sin^2\theta}}\, d\theta \qquad\qquad \sin\theta = e^x \\ &= \int \cot\theta\, d\theta \\ &= -\ln|\csc\theta + \tan\theta|\Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}} \\ &= \ln(1+\sqrt{2}) \end{aligned}$$

Likewise, if $y = \cos^{-1}(e^x)$ then $x = \ln(\cos y)$ so $x' = -\tan y$ and $1 + x'^2 = 1 + \tan^2 y = \sec^2 y$. Thus, the arc length is

$$\begin{aligned} \int_{\frac{\pi}{4}}^{0} \sqrt{\sec^2 y}\, dy &= -\int_0^{\frac{\pi}{4}} \sec y\, dy \\ &= -\ln|\sec y + \tan y|\Big|_0^{\frac{\pi}{4}} \\ &= \ln(1+\sqrt{2}) \end{aligned}$$

18. Assume a spring at res is 7 ft long. If 25 ft-lbs of work is required to hold the spring at a length of 10 ft, how much work is required to stretch the spring from 9 to 11 ft?

We have that $25 = \int_0^3 kx\ dx$ by Hooke's Law. Integrating and solving for k gives that $k = \frac{50}{9}$. We then evaluate $\int_2^4 \frac{50}{9}x\ dx = \frac{100}{3}$.

19. Ants excavate a chamber that is underground that is described as follows: Let S be the region of the fourth quadrant enclosed by $y = -\sqrt{x}$, $y = -1$, and the $y-$axis; revolve S about the $y-$axis. (a) Set up and simplify an integral for the volume of the chamber. (b) Suppose the chamber contained soil weighing 50 lb/ft^3. How much work did the ants do in raising the soil to ground level?

(a) We use disks with a radius of y^2 so the volume is $\pi \int_{-1}^{0} y^4\ dy = \frac{\pi}{5}$.

(b) Using the same setup from (a), a slice of soil has area πy^4 and a height of Δy so the force on the slice is $50\pi y^4 \Delta y$. Thus, the work to move the slice y feet is $50\pi y^5 \Delta y$. Hence, the total work is $\int_0^1 50\pi y^5\ dy = \frac{25\pi}{3}$.

20. A 5 lb bucket containing 10 lb of water is hanging at the end of a 300 ft rope which weighs 15 lbs. The other end of the rope is attached to a pulley. The rope is wound onto the pulley at a rate of 3 ft/s, causing the bucket to be lifted. Find the work done in lifting the bucket 200 ft if the water leaks out of the bucket at a rate of 1/4 lb/s.

At a height of x ft, the force on the rope is $\frac{1}{20}(300 - x)$ and the force on the water is $10 - \frac{1}{12}x$. Adding in the force on the bucket, the total force at x is $30 - \frac{2}{15}x$ so the total work is $\int_0^{200} 30 - \frac{2}{15}x\ dx = \frac{10000}{3}$.