Questions for recitation 31 March 2021

1. Consider the alternating p-series
$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^p}.$$
For what values of $p \in \mathbb{R}$ is this infinite series divergent, conditionally convergent, or absolutely convergent?

   Solution:
$$\sum a_n \begin{cases} \text{absolutely converges for } p > 1 \text{ by p-series} \\ \text{conditionally converges for } 0 < p \leq 1 \text{ by AST} \\ \text{diverges for } p \leq 0 \text{ by test for divergence} \end{cases}$$

2. Determine if each of the series below converges (absolutely/conditionally) or diverges. If possible, for each convergent series, determine the sum of the series. Be sure to fully motivate your answers.

   (a) $\displaystyle\sum_{n=3}^{\infty} \frac{\ln n}{\ln(\ln n)}$

   (b) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n 3^n}{n^n}$

   (c) $\displaystyle\sum_{n=1}^{\infty} \frac{(2n)!}{n!n!}$

   (d) $\displaystyle\sum_{n=1}^{\infty} \frac{n!n!}{(2n)!}$

   (e) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+2n}}{(\arctan(n))^n}$

   (f) $\displaystyle\sum_{n=1}^{\infty} \frac{\ln(n^2)3^n n!}{n^n}$

   (g) $\displaystyle\sum_{n=1}^{\infty} \frac{\sin(n)}{n^2}$

   Solution:

   (a) $n > \ln(n) \implies \ln(n) > \ln(\ln(n))$, so this diverges by the test for divergence.

   (b) The root test is an excellent choice when n is in exponents: $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \frac{6}{n} \to 0$, so this converges absolutely.

   (c) The ratio test is an excellent choice when n appears in factorials:
$$\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = \lim_{n\to\infty} |\frac{(2n+2)!}{2n!}||\frac{n!n!}{(n+1)!(n+1)!}| = \lim_{n\to\infty} \frac{(2n+1)(2n+2)}{(n+1)(n+1)} \to 4,$$
   so this diverges.

   (d) By an identical ratio test to (c), $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| \to \frac{1}{4}$ which implies absolute convergence.

   (e) Root test: $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \frac{1}{\tan^{-1}(n)} = \frac{2}{\pi}$, so this converges absolutely.

(f) Ratio test, (already grouping terms):

$$\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = \lim_{n\to\infty} \frac{\ln(n+1)^2}{\ln(n^2)} \frac{3^{n+1}}{3^n} \frac{(n+1)!}{n!} \frac{n^n}{(n+1)^{n+1}} = \lim_{n\to\infty} 1 \cdot 3 \cdot \frac{n+1}{n+1} \left(\frac{n}{n+1}\right)^n \to \frac{3}{e} > 1$$

So (whew!) this diverges by the ratio test. The $n \cdot e$ coming from the ratio test on n^n is something to keep in mind.

(g) Importantly, note that this does *not* satisfy the criteria of the alternating series test. However, noting that $0 \leq |a_n| \leq \frac{1}{n^2}$, this converges absolutely by direct comparison.

3. Do the following series converge or diverge? If they converge, is it conditional or absolute?

(a) $\sum_{n=1}^{\infty} (-1)^{n+1} \left(\frac{1+n}{n^2}\right)$

(b) $\sum_{n=1}^{\infty} (-1)^{n+1} \left(\frac{\ln(n)}{\ln(n^2)}\right)^{2n}$

(c) $\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{n^{4/5}}$

(d) $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{\cos(n\pi)}{n^{4/5}}$

(e) $\sum_{n=1}^{\infty} \frac{n^n}{5^n}$

Solution:

a) $\sum |a_n| = \sum \frac{1+n}{n^2}$, which diverges by limit comparison to $b_n = \frac{1}{n}$, so $\sum a_n$ does not converge absolutely. However, the series is alternating, and $|a_n|$ is decreasing, $|a_n| \to 0$, so $\sum a_n$ converges by the alternating series test. Since this converge is not absolute, it is conditional.

b)

$$\sum_{n=1}^{\infty} (-1)^{n+1} \left(\frac{\ln(n)}{\ln(n^2)}\right)^{2n} = \sum_{n=1}^{\infty} (-1)^{n+1} (\frac{1}{2})^{2n} = \sum_{n=1}^{\infty} -(\frac{-1}{4})^n \overset{geo}{=} \frac{1/4}{1+(1/4)} = 1/5.$$

This series converges absolutely.

c) $\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{n^{4/5}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^{4/5}}$ which has the same behavior as the series in part a (converges by alternating series, but the convergence is not absolute by p-series).

d) $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{(-1)^n}{n^{4/5}} = \sum_{n=1}^{\infty} \frac{-1}{n^{4/5}}$. This series diverges by the p-test ($p = 4/5 < 1$).

e) By the root test, $|a_n|^{1/n} = \frac{n}{5} \to \infty$, so this diverges. (Equivalently, the test for divergence works, as $a_n > 1$ for $n > 5$.)

4. State a criteria under which

$$\sum_{j=1}^{\infty}\left(\sum_{i=0}^{\infty} ax^i\right)^j$$

converges.

Solution: The inner sum is geometric, so:

$$\sum_{j=1}^{\infty}\left(\sum_{i=0}^{\infty} ax^i\right)^j = \sum_{j=1}^{\infty}\left(\frac{a}{1-x}\right)^j$$

if $|x| < 1$. But this outer sum is geometric too — and it will converge to

$$\frac{\frac{a}{1-x}}{1-\left(\frac{a}{1-x}\right)}$$

as long as $|\frac{a}{1-x}| < 1$. This is our second condition for convergence (with $|x| < 1$).