Alternating Series

Friday, March 26, 2021

1. Does the following series converge? Why or why not? If it does converge, for what value of m will the mth sum be within .001 of the infinite sum? $\sum_{n=1}^{\infty} \frac{(-1)^n}{3n}$

$b_n = \frac{1}{3n} > 0$ is decreasing and $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{3n} = 0$, so the series converges by Alternating series Test.

Using the Alternating Series Estimation Theorem, we are looking for m such that

$$|R_m| = |s - s_m| \leq b_{m+1} < 0.001$$

$$\Rightarrow \quad b_{m+1} = \frac{1}{3(m+1)} < 0.001$$

$$\Rightarrow \quad m+1 > \frac{1}{0.003}$$

$$\Rightarrow \quad m > \frac{1}{0.003} - 1$$

$$\Rightarrow \quad m > 332.3333$$

Since m is an integer, $m = 333$ suffices.

i.e.;

$$R_n = \sum_{n=1}^{\infty} \frac{(-1)^n}{3n} - \sum_{n=1}^{333} \frac{(-1)^n}{3n} < 0.001.$$

So, the approximation

$$\sum^{\infty} \frac{(-1)^n}{3n} \approx \sum_{n=1}^{333} \frac{(-1)^n}{3n}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{3n} \approx \sum_{n=1}^{333} \frac{(-1)^n}{3n}$$

is accurate to 2 decimal places.

2. Which of the following alternating series converge and which diverge? Be sure to fully support your answer.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$

$b_n = \frac{1}{n}$ is positive, decreasing and $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{n} = 0.$

So $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges by the Alternating Series Test.

(b) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{10^n}{n^{10}}$

$b_n = \frac{10^n}{n^{10}}$ is positive but not decreasing since 10^n (exponential) grows faster than n^{10} (polynomial). In fact

$$\lim_{n\to\infty} \frac{10^n}{n^{10}} = \infty$$ (and you can check this by L'Hospital's rule)

So $a_n = (-1)^n \frac{10^n}{n^{10}} \not\to 0.$

Hence, $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{10^n}{n^{10}}$ diverges by Test for Divergence.

(c) $\sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{\ln(n)}$

$b_n = \frac{1}{\ln n}$ positive, decreasing and $\lim_{n\to\infty} \frac{1}{\ln n} = 0$.

So $\sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{\ln n}$ is convergent by the Alternating Series Test.

(d) $\sum_{n=2}^{\infty} (-1)^{n+1} \frac{\ln(n)}{\ln(n^2)}$ $= \sum_{n=2}^{\infty} (-1)^{n+1} \frac{\ln n}{2 \ln n}$ (by property of logarithm)

$$= \sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{2}$$

$\Rightarrow b_n = \frac{1}{2}$ is positive and satisfies $b_{n+1} \leq b_n$ for all $n \geq 2$.

However,

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{2} = 1/2 \neq 0$$

So $a_n = (-1)^{n+1} \frac{\ln n}{\ln n^2} \not\longrightarrow 0$

Hence, $\sum_{n=2}^{\infty} (-1)^{n+1} \frac{\ln n}{\ln n^2}$ diverges by Test for Divergence

(e) $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sqrt{n}+1}{n+1}$

$b_n = \dfrac{\sqrt{n}+1}{n+1}$ is positive and decreasing (since $\sqrt{n} < n$). Also (by the same reason or using L'Hospital's rule),

$$\lim_{n\to\infty} \frac{\sqrt{n}+1}{n+1} = 0.$$

So $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sqrt{n}+1}{n+1}$

converges by the Alternating Series Test.

(f) $\displaystyle\sum_{n=1}^{\infty} n^{\frac{-1}{10}}\cos(n\pi)$ $= \displaystyle\sum_{n=1}^{\infty} n^{-1/10}(-1)^n$ since $\cos(n\pi) = (-1)^n$.

So $b_n = n^{-1/10} = \dfrac{1}{n^{1/10}}$ and this is positive and decreasing. Also,

$$\lim_{n\to\infty} \frac{1}{n^{1/10}} = 0$$ (very slowly but will surely $\longrightarrow 0$)

Hence, $\displaystyle\sum_{n=1}^{\infty} n^{-1/10}\cos(n\pi)$ converges by the Alternating Series Test.