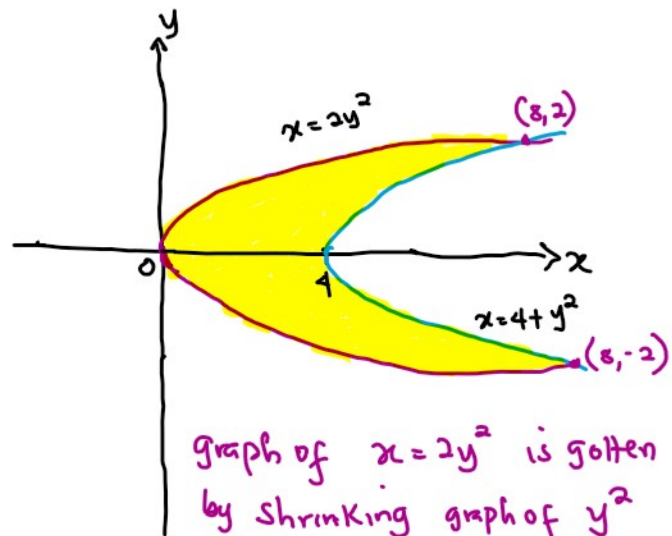
Areas Between Curves--#17, 20, 27, 36

Tuesday, February 9, 2021

17. $x = 2y^2, \quad x = 4 + y^2$



graph of $x = 2y^2$ is gotten by shrinking graph of y^2 vertically and $x = 4+y^2$ by shifting y^2 horizontally

We first find the intersection points of the two curves by solving

$$x = 2y^2 = 4 + y^2$$

$$\Rightarrow \quad y^2 = 4$$

$$\Rightarrow \quad y = \pm 2.$$

$$\Rightarrow \quad x = 2(\pm 2)^2 = 8$$

Thus,

$$\text{Area enclosed} = \int_{-2}^{2} \left(4 + y^2 - 2y^2\right) dy$$

$$= \int_{-2}^{2} (4 - y^2)\, dy$$

$$= 4y - \frac{y^3}{3} \Big|_{-2}^{2}$$

$$= 4(2) - \frac{2^3}{3} - \left(4(-2) - \frac{(-2)^3}{3}\right)$$

$$= 8 - \frac{8}{3} + 8 - \frac{8}{3}$$

$$= \frac{32}{3}$$

20. $x = y^4, \quad y = \sqrt{2 - x}, \quad y = 0$

$y = \sqrt{2-x} \Rightarrow y^2 = 2 - x \Rightarrow x = 2 - y^2$

$y = \sqrt{2-x} \Rightarrow y^2 = 2-x \Rightarrow x = 2-y^2$

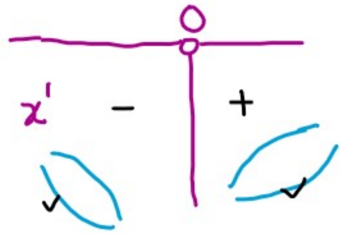
$x = y^4$

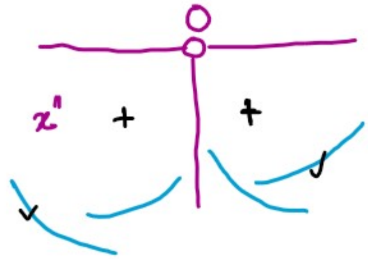
$x' = 4y^3 = 0 \Rightarrow y = 0$

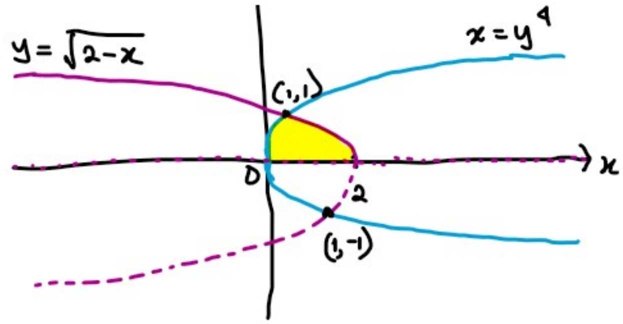
Also,

$x'' = 12y^2 = 0 \Rightarrow y = 0$

Furthermore, $x \longrightarrow \infty$ as $y \longrightarrow \pm\infty$







We now solve for the intersections of the two curves

$$x = y^4 = 2 - y^2$$

$$\Rightarrow y^4 + y^2 - 2 = 0$$

$$\Rightarrow u^2 + u - 2 = 0 \quad \text{if } u = y^2$$

$$\Rightarrow (u-1)(u-1) = 0$$

$$\Rightarrow (u-1)^2 = 0$$

$$\Rightarrow (y^2-1)^2 = 0$$

$$\Rightarrow (y-1)^2(y+1)^2 = 0$$

$$\Rightarrow y = -1, 1$$

$$\text{So} \quad x = 2-(\pm 1)^2 = 2-1 = 1$$

Thus,

$$\text{Area enclosed} = \int_0^1 (2-y^2-y^4)\,dy$$

$$= 2y - \frac{y^3}{3} - \frac{y^5}{5}\Big|_0^1$$

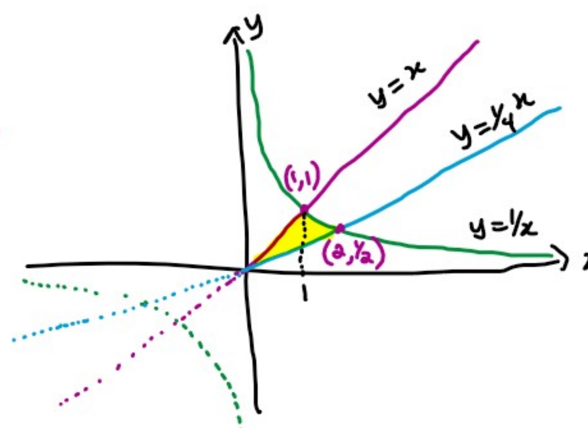
$$= 2y - \frac{y^3}{3} - \frac{y^5}{5} \Big|_0$$

$$= 2(1) - \frac{1^3}{3} - \frac{1^5}{5} - \left(2(0) - \frac{0^3}{3} - \frac{0^5}{5}\right)$$

$$= 2 - \frac{1}{3} - \frac{1}{5}$$

$$= \frac{22}{15}$$

27. $y = 1/x, \quad y = x, \quad y = \frac{1}{4}x, \quad x > 0$



Notice that $x = 0$ is a vertical asymptote for $y = 1/x$ and $y = 0$ is a horizontal asymptote for $y = 1/x$.

Also,

$y = 1/x = x \Rightarrow 1 = x^2 \Rightarrow x = \pm 1 \Rightarrow y = \pm 1$

and

$y = 1/x = 1/4\, x \Rightarrow 4 = x^2 \Rightarrow x = \pm 2 \Rightarrow y = \pm 1/2$

graph of $y = 1/4\, x$ is a vertical shrink of the graph of $y = x$.

So

$$\text{Area enclosed} = \int_0^1 \left(x - \tfrac{1}{4}x\right)dx + \int_1^2 \left(\tfrac{1}{x} - \tfrac{1}{4}x\right)dx$$

$$= \frac{3}{4}\int_0^1 x\, dx + \int_1^2 \left(\frac{1}{x} - \frac{1}{4}x\right)dx$$

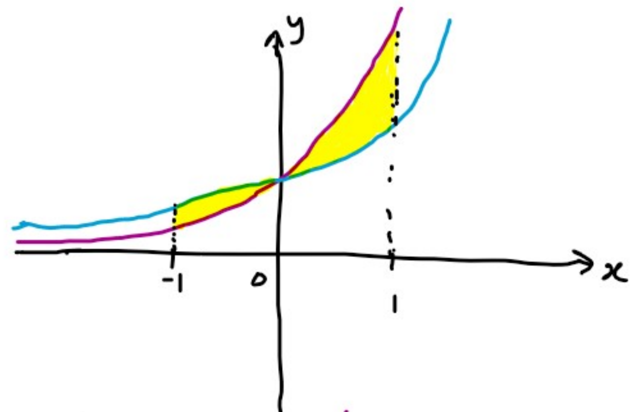
$$= \frac{3}{4} \cdot \frac{x^2}{2}\Big|_0^1 + \ln x - \frac{1}{4} \cdot \frac{x^2}{2}\Big|_1^2$$

$$= \frac{3}{8}\left(1^2 - 0^2\right) + \ln 2 - \frac{2^2}{8} - \left(\ln 1 - \frac{1^2}{8}\right)$$

$$= \frac{3}{8} + \ln 2 - \frac{4}{8} + \frac{1}{8}$$

$$= \ln 2$$

36. $\int_{-1}^{1} |\,3^x - 2^x\,|\, dx$



$$\int_{-1}^{1} |3^x - 2^x|\,dx = \int_{-1}^{0} (2^x - 3^x)\,dx + \int_{0}^{1} (3^x - 2^x)\,dx$$

$$= \left(\frac{2^x}{\ln 2} - \frac{3^x}{\ln 3}\right)\Bigg|_{-1}^{0} + \left(\frac{3^x}{\ln 3} - \frac{2^x}{\ln 2}\right)\Bigg|_{0}^{1}$$

$$= \left(\left(\frac{2^0}{\ln 2} - \frac{3^0}{\ln 3}\right) - \left(\frac{2^{-1}}{\ln 2} - \frac{3^{-1}}{\ln 3}\right)\right) + \left(\left(\frac{3^1}{\ln 3} - \frac{2^1}{\ln 2}\right) - \left(\frac{3^0}{\ln 3} - \frac{2^0}{\ln 2}\right)\right)$$

$$= \left(\frac{1}{\ln 2} - \frac{1}{\ln 3} - \frac{1}{2\ln 2} + \frac{1}{3\ln 3}\right) + \left(\frac{3}{\ln 3} - \frac{2}{\ln 2} - \frac{1}{\ln 3} + \frac{1}{\ln 2}\right)$$

$$= \left(\frac{1}{2\ln 2} - \frac{2}{3\ln 3}\right) + \left(\frac{2}{\ln 3} - \frac{1}{\ln 2}\right)$$

$$\frac{2-1}{2\ln 2}$$

$$\frac{1}{3\ln 3} - \frac{3}{3\ln 3} = \frac{-2}{3\ln 3}$$

$$= \frac{1}{2\ln 2} - \frac{2}{3\ln 3} + \frac{2}{\ln 3} - \frac{1}{\ln 2}$$

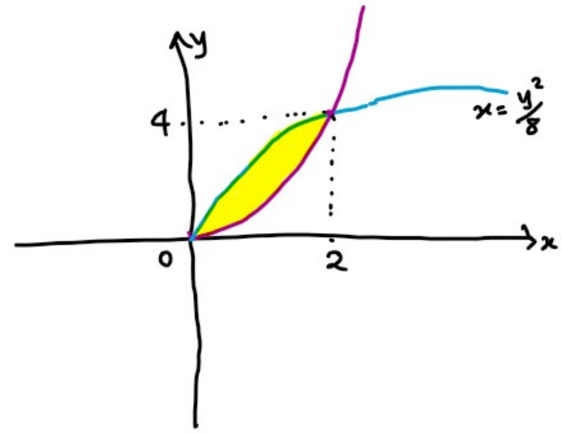
$$= \frac{1}{2\ln 2} - \frac{2}{3\ln 3} + \frac{3\cdot 2}{3\ln 2} - \frac{2\cdot 1}{2\ln 2}$$

$$= \frac{4}{3\ln 2} - \frac{1}{2\ln 2}$$

1. Find the area of the region in the first quadrant bounded by the curves $x = \sqrt{y}$ and $x = \frac{y^2}{8}$.

$x = \sqrt{y}$

$x = \sqrt{y} \Rightarrow y = x^2, \; x \geq 0$



Solving for the intersections of the two curves:

$$\sqrt{y} = \frac{y^2}{8}$$

$$\Rightarrow y = \frac{y^4}{64}$$

$$\Rightarrow y^4 - 64y = 0$$

$$\Rightarrow y(y^3 - 64) = 0$$

$$\Rightarrow y = 0 \text{ or } y^3 - 64 = 0$$

$$\Rightarrow y = 0 \text{ or } y = 4$$

So $y = 0 \Rightarrow x = 0$ and $y = 4 \Rightarrow x = 2$.

Thus,

$$\text{Area of region} = \int_0^4 \left(\sqrt{y} - \frac{y^2}{8}\right) dy$$

$$= \int_0^4 \left(y^{1/2} - \frac{y^2}{8}\right) dy$$

$$= \frac{2}{3} y^{3/2} - \frac{y^3}{8 \cdot 3} \Big|_0^4$$

$$= \frac{2}{3}\left(\sqrt{4}\right)^3 - \frac{4^3}{8 \cdot 3} - \left(\frac{2}{3}\left(\sqrt{0}\right)^3 - \frac{0^3}{8 \cdot 3}\right)$$

$$= \frac{2}{3}(8) - \frac{64}{8 \cdot 3}$$

$$= \frac{16}{3} - \frac{8}{3}$$

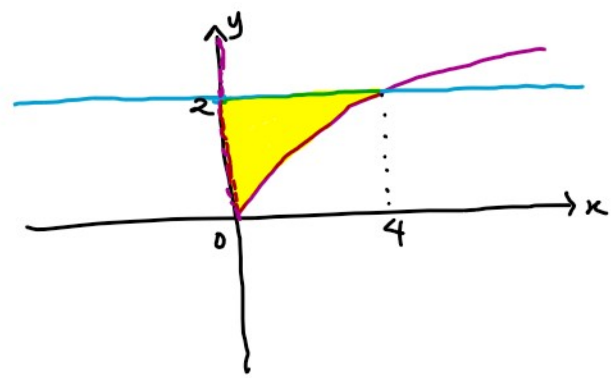
$$= \frac{8}{3}$$

2. Find the area of the region bounded by the lines $y = 2$, $x = 0$, and the curve $y = \sqrt{x}$.

$y = \sqrt{x} \Rightarrow x = y^2,\ y \geq 0$

When $y = 2$ and $y = \sqrt{x}$ intersect,

$$2 = \sqrt{x} \Rightarrow x = 4$$



So

$$\text{Area of region} = \int_0^4 \left(2 - \sqrt{x}\right) dx$$

$$= \int_0^4 \left(2 - x^{1/2}\right) dx$$

$$= 2x - \frac{2}{3} x^{3/2} \Big|_0^4$$

$$= 2(4) - \frac{2}{3}\left(4^{3/2}\right) - \left(2(0) - \frac{2}{3}\left(0^{3/2}\right)\right)$$

$$= 8 - \frac{2}{3}(8)$$

$$= \frac{24 - 16}{3}$$

$$= \frac{8}{3}$$

3. Find the area of the region bounded by $y = x - x^3$ and $y = x^2 - x$ for $x \geq 0$.

The curves intersect when:

$y = x - x^3 \Rightarrow y' = 1 - 3x^2 = 0$

$\Rightarrow x^2 = 1/3$

$y = x - x^3 = x^2 - x$

The curves intersect when:

$$y = x - x^3 = x^2 - x$$

$$\Rightarrow \quad x^3 + x^2 - 2x = 0$$

$$\Rightarrow \quad x(x^2 + x - 2) = 0$$

$$\Rightarrow \quad x(x-1)^2 = 0$$

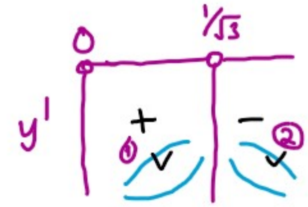
$$\Rightarrow \quad x = 0, \; x = 1.$$

Also $x - x^3 = 0 \Rightarrow x = -1, 0, 1$

$$y = x - x^3 \Rightarrow y' = 1 - 3x^2 = 0$$

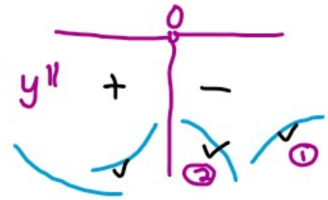
$$\Rightarrow x^2 = 1/3$$

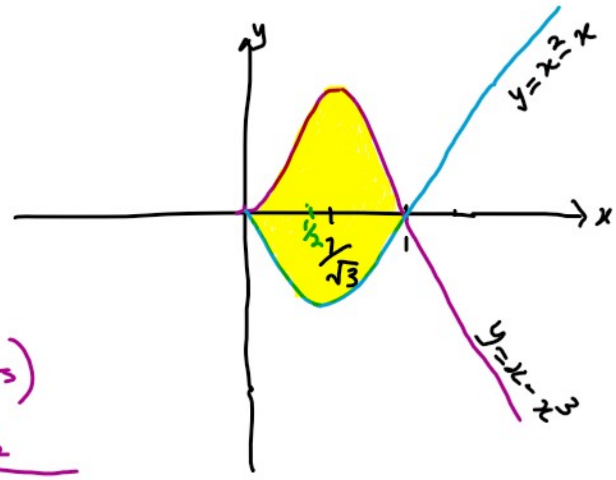
$$\Rightarrow x = \frac{1}{\sqrt{3}} \text{ since } x \geq 0$$



$$y'' = -6x = 0 \Rightarrow x = 0$$

Also, $y \to -\infty$ as $x \to \infty$

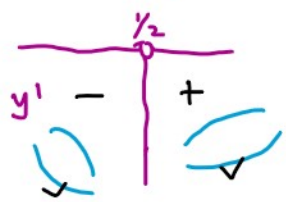




On the other hand,

$y = x^2 - x \Rightarrow x(x-1) = 0 \Rightarrow x = 0, 1$ (Intercepts)

$y' = 2x - 1 = 0 \Rightarrow x = 1/2.$



Also,

$y'' = 2$ ($\Rightarrow y''$ is $+$ and so $\smile$)

Furthermore, $y = x^2 - x \to \infty$ as $x \to \infty$

Notice that by completing the squares:

$$\begin{aligned} y &= x^2 - x \\ &= x^2 - x + \left(-\tfrac{1}{2}\right)^2 - \left(-\tfrac{1}{2}\right)^2 \\ &= (x - 1/2)^2 - (-1/2)^2 \\ &= (x - 1/2)^2 - \tfrac{1}{4} \end{aligned}$$

(so graph by horizontal and vertical shift of x^2)

Thus,

Thus,

$$\text{Area of the region} = \int_0^1 \left[(x - x^3) - (x^2 - x)\right] dx$$

$$= \int_0^1 (2x - x^2 - x^3)\,dx$$

$$= x^2 - \frac{x^3}{3} - \frac{x^4}{4} \Big|_0^1$$

$$= \left(1^2 - \frac{1^3}{3} - \frac{1^4}{4}\right) - \left(0^2 - \frac{0^3}{3} - \frac{0^4}{4}\right)$$

$$= 1 - \frac{1}{3} - \frac{1}{4}$$

$$= \underline{\underline{\frac{5}{12}}}$$