Average Value of a function-6.5: #6, 12, 14, 15, 18

Thursday, February 25, 2021

(6) $f(x) = \dfrac{x^2}{(x^3+3)^2}$, $[-1,1]$

Average value of the function on $[-1,1]$:

$$f_{ave} = \frac{1}{1-(-1)} \int_{-1}^{1} \frac{x^2}{(x^3+3)^2}\,dx$$

Let $u = x^3 + 3$. Then $\dfrac{du}{dx} = 3x^2 \Rightarrow \dfrac{du}{3x^2} = dx$

Also, $x = 1 \Rightarrow u = 1^3 + 3 = 4$

$x = -1 \Rightarrow u = (-1)^3 + 3 = 2$

$$= \frac{1}{2} \int_{2}^{4} \frac{x^2}{u^2} \cdot \frac{du}{3x^2}$$

$$= \frac{1}{2\cdot 3} \int_{2}^{4} \frac{1}{u^2}\,du$$

$$= \frac{1}{6} \left(-\frac{1}{u} \right) \Big|_{2}^{4}$$

$$= \frac{1}{6} \left(-\frac{1}{4} + \frac{1}{2} \right)$$

$$= \frac{1}{24}$$

(12) $f(x) = 2xe^{-x^2}$, $[0,2]$.

(a) $$f_{ave} = \frac{1}{2-0} \int_{0}^{2} 2xe^{-x^2}\,dx$$

Let $u = -x^2$. Then $\dfrac{du}{dx} = -2x \Rightarrow \dfrac{du}{-2x} = dx$

$$= \frac{1}{2}\int_{x=0}^{x=2} 2x \cdot e^{u} \cdot \frac{du}{-2x}$$

$$= -\frac{1}{2}\int_{0}^{2} e^{u}\,du$$

$$= -\frac{1}{2}e^{u}\Big|_{x=0}^{x=2}$$

$$= -\frac{1}{2}e^{-x^2}\Big|_{0}^{2}$$

$$= -\frac{1}{2}e^{-2^2} + e^{-0^2}$$

$$= \frac{1}{2}(1 - e^{-4})$$

(b) $f_{ave} = f(c) \rightarrow c = ?$

$$f(c) = 2xe^{-x^2}\Big|_{x=c} = \frac{1}{2}(1-e^{-4}) \text{ from (a)}$$

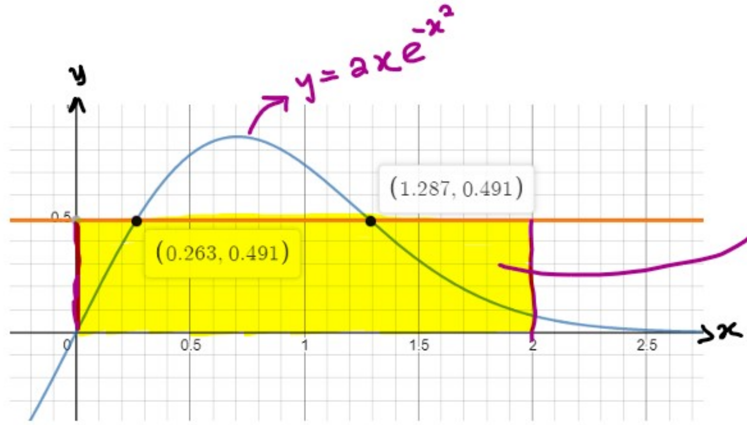
$$\Rightarrow \quad 2ce^{-c^2} = \frac{1}{2}(1-e^{-4})$$

$$\Rightarrow \quad 4ce^{-c^2} + e^{-4} - 1 = 0$$

(Plot the graph of $g(x) = 4xe^{-x^2} + e^{-4} - 1$ and read off x-intercepts)

$$\Rightarrow \quad c = 0.263 \text{ or } c = 1.287$$

(c)



Rectangle whose area equals area under the graph of $y = 2xe^{-x^2}$ on $[0, 2]$.

(14) $f(x) = 2 + 6x - 3x^2$ on $[0, b]$, $f_{ave} = 3$, $b = ?$

$$3 = f_{ave} = \frac{1}{b-0}\int_0^b (2 + 6x - 3x^2)\,dx$$

$$= \frac{1}{b}\left(2x + 3x^2 - x^3\right)\Big|_0^b$$

$$= \frac{1}{b}\left(2b + 3b^2 - b^3\right) - \frac{1}{b}\left(2(0) + 3(0^2) - 0^3\right)$$

$$= \frac{1}{b}\cdot b\left(2 + 3b - b^2\right)$$

$$= 2 + 3b - b^2$$

$$\Rightarrow 3 - 2 - 3b + b^2 = 0$$

$$\Rightarrow b^2 - 3b + 1 = 0$$

$$\Rightarrow b = \frac{3 \pm \sqrt{(-3)^2 - 4(1)(1)}}{2(1)}$$

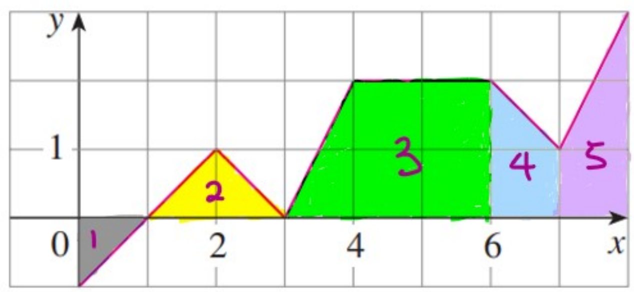
2(1)

$$= \frac{3 \pm \sqrt{9-4}}{2}$$

$$= \frac{3 \pm \sqrt{5}}{2}$$

So $b = \frac{3+\sqrt{5}}{2}$ or $b = \frac{3-\sqrt{5}}{2}$ since both are positive.

(15) $f_{ave} = ?$ on $[0,8]$.



$$f_{ave} = \frac{1}{8-0}\int_0^8 f(x)\,dx$$

$$= \frac{1}{8}\left(\text{sum of areas bounded by } f \text{ and the } x\text{-axis}\right)$$

$$= \frac{1}{8}\left(-\text{area } 1 + \text{area } 2 + \text{area } 3 + \text{area } 4 + \text{area } 5\right)$$

$$= \frac{1}{8}\left(-\left(\tfrac{1}{2}\times 1\times 1\right) + \left(\tfrac{1}{2}\times 2\times 1\right) + \left(\tfrac{1}{2}(2+3)\times 2\right) + \left(\tfrac{1}{2}(2+1)\times 1\right) + \left(\tfrac{1}{2}(3+1)\times 1\right)\right)$$

$$= \frac{1}{8}\left(-\frac{1}{2} + 1 + 5 + \frac{3}{2} + 2\right)$$

$$= \frac{9}{8}$$

(18) $v(r) = \frac{P}{4\eta l}(R^2 - r^2)$ on $[0, R]$, $f_{ave} = ?$

$$V_{ave} = \frac{1}{R-0}\int_0^R \frac{P}{4\eta l}(R^2 - r^2)\,dr$$

$$= \frac{P}{4\eta R l}\int_0^R (R^2 - r^2)\,dr$$

$$= \frac{P}{4\eta R l}\left(R^2 r - \frac{r^3}{3}\right)\Big|_0^R$$

$$= \frac{P}{4\eta R l}\left(R^2 \cdot R - \frac{R^3}{3}\right) - \frac{P}{4\eta R l}\left(R^2 \cdot 0 - \frac{0^3}{3}\right)$$

$$= \frac{P}{4\eta R l}\left(\frac{3R^3 - R^3}{3}\right)$$

$$= \frac{P}{4\eta R l}\left(\frac{2R^3}{3}\right)$$

$$= \frac{PR^3}{6\eta R l}$$

$$= \frac{PR^2}{6\eta l}$$

Also,

$$v'(r) = \frac{P}{2\eta l}(0-2r)$$

$$= -\frac{P}{2\eta l}r$$

$$\Rightarrow \quad v''(r) = -\frac{P}{2\eta l} < 0$$ assuming all constants are positive.

$\Rightarrow$ v is concave downward everywhere

$\Rightarrow$ V_{max} occurs when $v'(r) = 0$

$$\Rightarrow -\frac{P}{2\eta l}r = 0$$

$$\Rightarrow r = 0$$

So

$$V_{max}(0) = \frac{P}{4\eta l}(R^2 - 0^2) = \frac{PR^2}{4\eta l}$$

Clearly, V_{ave} and V_{max} differ by a constant multiple.

More precisely,

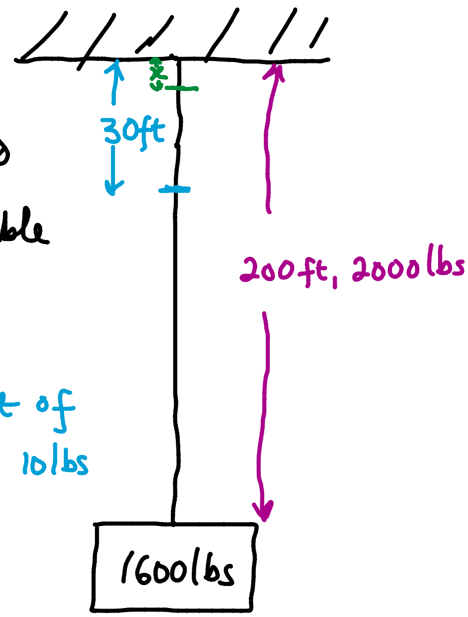
$$V_{max} = \frac{PR^2}{4\eta l}$$

$$= \frac{1}{4}\left(\frac{6PR^2}{6\eta l}\right)$$

$$= \frac{6}{4}\left(\frac{PR^2}{6\eta l}\right)$$

i.e; $\quad V_{max} = \frac{3}{2} V_{ave}$.

3. A 1600 lb elevator is suspended by a 200 ft cable that weighs 2000 lbs. How much work is required to lift the elevator from the basement to the third floor (30 ft)?



If we divide the 30ft distance into infinitesimal length, $x$ft, then the work required to be done per $x$ft is raising $(200-x)$ft of cable and 1600lbs elevator.

Thus,

force, $f(x) = 1600 + (200-x)10$ since each ft of cable weighs 10lbs

Hence,

$$\text{Work required} = \int_0^{30} f(x)\,dx$$

$$= \int_0^{30} \left(1600 + (200-x)10\right) dx$$

$$= \int_0^{30} (3600 - 10x)\,dx$$

$$= 3600x - 5x^2 \Big|_0^{30}$$

$$= 3600(30) - 5(30^2)$$

$$= 108000 - 4500$$

$$= 103500\text{ J}$$