Improper Integrals

Thursday, January 28, 2021 1:54 AM

1. Do the following integrals converge or diverge? If they converge, evaluate:

(a)

$$\int_{-\infty}^{0} \frac{1}{2x-5}\, dx$$

$$\int_{-\infty}^{0} \frac{1}{2x-5}\, dx = \lim_{t \to -\infty} \int_{t}^{0} \frac{1}{2x-5}\, dx$$

$$= \lim_{t \to -\infty} \frac{1}{2} \int_{t}^{0} \frac{2}{2x-5}\, d$$

$$= \frac{1}{2} \lim_{t \to -\infty} \ln|2x-5| \Big|_{t}^{0}$$

$$= \frac{1}{2} \lim_{t \to -\infty} \left(\ln 5 - \ln|2t-5| \right)$$

$$= \frac{1}{2} \ln 5 - \lim_{t \to -\infty} \ln|2t-5|$$

$$= \frac{1}{2} \ln 5 - \infty$$

$$= -\infty$$

Hence, $\int_{-\infty}^{0} \frac{1}{2x-5}\, dx$ diverges.

(b)

$$\int_0^1 \frac{1}{4y-1}\,dy$$

$y = 1/4$

$\lim_{y \to 1/4^-} \frac{1}{4y-1} = -\infty$

$\lim_{y \to 1/4^+} \frac{1}{4y-1} = \infty$

So the integrand $\frac{1}{4y-1}$ has an infinite discontinuity at $y = \frac{1}{4}$.

Thus,

$$\int_0^1 \frac{1}{4y-1}\,dy = \lim_{t \to \frac{1}{4}^-} \int_0^t \frac{1}{4y-1}\,dy + \lim_{t \to \frac{1}{4}^+} \int_t^1 \frac{1}{4y-1}\,dy$$

if both limits exist as finite numbers.

But

$$\lim_{t \to \frac{1}{4}^-} \int_0^t \frac{1}{4y-1}\,dy = \lim_{t \to \frac{1}{4}^-} \frac{1}{4} \int_0^t \frac{4}{4y-1}\,dy$$

$$= \frac{1}{4} \lim_{t \to \frac{1}{4}^-} \ln|4y-1| \Big|_0^t$$

$$= \frac{1}{4} \lim_{t \to \frac{1}{4}^-} \Big(\ln|4t-1| - \ln|4(0)-1| \Big)$$

$$= \frac{1}{4} \lim_{t \to \frac{1}{4}^-} \ln|4t-1| - \frac{1}{4}\ln 3$$

$$= -\infty + \frac{1}{4}\ln 3$$

$$= -\infty.$$

Hence, $\int_0^1 \frac{1}{4y-1}\,dy$ is divergent.

(c)

$$\int_0^1 \frac{\ln x}{\sqrt{x}}\, dx$$

$\lim_{x\to 0^+} \frac{\ln x}{\sqrt{x}} = -\infty$

$\Rightarrow x=0$ is an infinite discontinuity point.

Thus,

$$\int_0^1 \frac{\ln x}{\sqrt{x}}\,dx = \lim_{t\to 0^+} \int_t^1 \frac{\ln x}{\sqrt{x}}\,dx$$

Let $u = \ln x$ and $dv = x^{-1/2}dx$

Then $du = \frac{1}{x}dx$ and $v = 2x^{1/2}$

$$= \lim_{t\to 0^+}\left(uv - \int_t^1 v\,du\right)$$

$$= \lim_{t\to 0^+}\left(2\sqrt{x}\ln x\Big|_t^1 - 2\int_t^1 x^{-1/2}dx\right)$$

$$= \lim_{t\to 0^+}\left(2\sqrt{x}\ln x\Big|_t^1 - 4\sqrt{x}\Big|_t^1\right)$$

$$= \lim_{t\to 0^+}\left(\left(2\sqrt{1}\ln 1 - 2\sqrt{t}\ln t\right) - 4\left(\sqrt{1}-\sqrt{t}\right)\right)$$

$$= \lim_{t\to 0^+}\left(-2\sqrt{t}\ln t - 4 + 4\sqrt{t}\right)$$

$$= 2\lim_{t\to 0^+}\frac{\ln t}{1/\sqrt{t}} - 4 + 4\lim_{t\to 0^+}\sqrt{t}$$

$$= 2 \lim_{t \to 0^+} \frac{1/t}{-1/2\sqrt{t^3}} - 4$$

$$= 2 \lim_{t \to 0^+} \left(-2t^{3/2-1}\right) - 4$$

$$= 2 \lim_{t \to 0^+} \left(-2\sqrt{t}\right) - 4$$

$$= 0 - 4$$

$$= -4$$

So, the integral converges and

$$\int_0^1 \frac{\ln x}{\sqrt{x}} dx = -4$$

2. Does $\int_0^\pi \frac{\sin x}{\sqrt{\pi - x}}\, dx$ converge or diverge? (Hint: $\sin(\pi - x) = \sin x$.)

$$\lim_{x \to \pi} \frac{\sin x}{\sqrt{\pi - x}} \overset{\text{L'Hospital}}{=} \lim_{x \to \pi} \frac{\cos x}{\frac{1}{2\sqrt{\pi - x}}} = \lim_{x \to \pi} 2\sqrt{\pi - x}\cos x = 0.$$

So $\int_0^\pi \frac{\sin x}{\sqrt{\pi - x}} dx$ is not an improper integral and thus converges.

Moreover, from the hint, $\sin x = \sin(\pi - x)$ and

$0 \le \sin(\pi - x) \le \pi - x$ on $[0, \pi]$.

$$\Rightarrow \quad \frac{\sin x}{\sqrt{\pi - x}} = \frac{\sin(\pi - x)}{\sqrt{\pi - x}} \le \frac{\pi - x}{\sqrt{\pi - x}} = \sqrt{\pi - x}$$

$$\Rightarrow \quad \int_0^{\pi} \frac{\sin x}{\sqrt{\pi - x}}\,dx \le \int_0^{\pi} \sqrt{\pi - x}\,dx = \int_0^{\pi} (\pi - x)^{1/2}\,dx$$

$$= -\frac{2}{3}(\pi - x)^{3/2}\Big|_0^{\pi}$$

$$= \frac{2}{3}\pi^{3/2}$$

Hence, $\int_0^{\pi} \frac{\sin x}{\sqrt{\pi - x}}\,dx$ is convergent by Comparison Theorem in the book.

3. (a) Evaluate $\int_0^\infty t^2 e^{-3t}\,dt$

$$\begin{array}{ccc} & u & dv \\ + & t^2 & e^{-3t} \\ - & 2t & \frac{e^{-3t}}{-3} \\ + & 2 & \frac{e^{-3t}}{9} \\ - & 0 & \frac{e^{-3t}}{-27} \end{array}$$

$$\int_0^\infty t^2 e^{-3t}\,dt = \lim_{x \to \infty} \int_0^x t^2 e^{-3t}\,dt$$

$$= \lim_{x \to \infty} \left(-\frac{t^2 e^{-3t}}{3} - \frac{2te^{-3t}}{9} - \frac{2e^{-3t}}{27} \right)\Big|_0^x$$

$$= \lim_{x \to \infty} \left(\left(-\frac{x^2 e^{-3x}}{3} - \frac{2xe^{-3x}}{9} - \frac{2e^{-3x}}{27} \right) - \left(-\frac{0^2 e^{-3(0)}}{3} - \frac{2(0)e^{-3(0)}}{9} - \frac{2e^{-3(0)}}{27} \right) \right)$$

$$= \frac{-1}{27} \lim_{x\to\infty} \left(\frac{9x^2 + 6x + 2}{e^{3x}} \right) + \frac{2}{27}$$

$$\overset{L'H.}{=} \frac{-1}{27} \lim_{x\to\infty} \left(\frac{18x + 6}{3e^{3x}} \right) + \frac{2}{27}$$

$$\overset{L'H.}{=} -\frac{1}{27} \lim_{x\to\infty} \left(\frac{18}{9e^{3x}} \right) + \frac{2}{27}$$

$$= -\frac{1}{27}(0) + \frac{2}{27}$$

$$= \frac{2}{27}$$

(b) It's very important in probability theory to construct functions that integrate to one. To do so, we often multiply the functions used in probability models by constants to "normalize" them so that they integrate correctly. Find the normalizing constant c such that

$$1 = \int_0^\infty ct^2e^{-3t}\,dt.$$

$$1 = \int_0^\infty ct^2e^{-3t}dt = c\int_0^\infty t^2e^{-3t}dt = \frac{2}{27}c \qquad \text{(By Part (a))}$$

$$\Rightarrow \quad c = \frac{27}{2}.$$

(c) Generalize your previous two answers for *any* $\lambda > 0$ and non-negative integer n. That is, find $\int_0^\infty t^n e^{-\lambda t}\,dt$, then find c such that

$$1 = \int_0^\infty c t^n e^{-\lambda t}\,dt.$$

(This is called the Gamma distribution, and plays a major role in determining how much time elapses before observing a specified event or events. The exponent n controls "how many" events we wait on while the rate constant λ determines "how often" those events occur.)

	u	dv
$+$	t^n	$e^{-\lambda t}$
$-$	nt^{n-1}	$\dfrac{e^{-\lambda t}}{-\lambda}$
$+$	$n(n-1)t^{n-2}$	$\dfrac{e^{-\lambda t}}{\lambda^2}$
$-$	$n(n-1)(n-2)t^{n-3}$	$\dfrac{e^{-\lambda t}}{-\lambda^3}$
	$\vdots$	$\vdots$
	$(-1)^k n(n-1)(n-2)\cdots(n-k)t^{n-k}$	$\dfrac{e^{-\lambda t}}{(-1)^k \lambda^k}$
	$\vdots$	$\vdots$
	$(-1)^{n-1} n(n-1)\cdots(n-k)\cdots 1\, t$	$\dfrac{e^{-\lambda t}}{(-1)^{n-1} \lambda^{n-1}}$
	$(-1)^n n(n-1)(n-2)\cdots 1$	$\dfrac{e^{-\lambda t}}{(-1)^n \lambda^n}$
	0	$\dfrac{e^{-\lambda t}}{(-1)^{n+1} \lambda^{n+1}}$

Thus,

$$\int^{\infty} t^n e^{-\lambda t}\,dt = \lim \int^{x} t^n e^{-\lambda t}\,dt$$

$$\int_0^\infty t^n e^{-\lambda t}\,dt = \lim_{x\to\infty}\int_0^x t^n e^{-\lambda t}\,dt$$

$$= \lim_{x\to\infty}\left(-\frac{t^n e^{-\lambda t}}{\lambda} - \frac{n t^{n-1} e^{-\lambda t}}{\lambda^2} - \frac{n(n-1)t^{n-2} e^{-\lambda t}}{\lambda^3} - \cdots - \frac{(n-1)(n-2)\cdots(n-k)t^{n-k} e^{-\lambda t}}{\lambda^{k+1}} - \cdots - \frac{n(n-1)(n-2)\cdots 1\, e^{-\lambda t}}{\lambda^{n+1}}\right)\Bigg|_0^x$$

$$= \lim_{x\to\infty}\left(-\frac{x^n e^{-\lambda x}}{\lambda} - \frac{n x^{n-1} e^{-\lambda x}}{\lambda^2} - \frac{n(n-1)\, x^{n-2} e^{-\lambda x}}{\lambda^3} - \cdots - \frac{(n-1)(n-2)\cdots(n-k)\, x^{n-k} e^{-\lambda x}}{\lambda^{k+1}} - \cdots - \frac{n(n-1)(n-2)\cdots 1\, e^{-\lambda x}}{\lambda^{n+1}}\right)$$

$$- \lim_{x\to\infty}\left(-\frac{0^n e^{-\lambda\cdot 0}}{\lambda} - \frac{n\cdot 0^{n-1}\cdot e^{-\lambda\cdot 0}}{\lambda^2} - \frac{n(n-1)\cdot 0^{n-2}\cdot e^{-\lambda\cdot 0}}{\lambda^3} - \cdots - \frac{(n-1)(n-2)\cdots(n-k)\cdot 0^{n-k} e^{-\lambda\cdot 0}}{\lambda^{k+1}} - \cdots - \frac{n(n-1)(n-2)\cdots 1\, e^{-\lambda\cdot 0}}{\lambda^{n+1}}\right)$$

Just like in part a, each term that contains an $e^{-\lambda x}$ goes to 0 as $x \to \infty$.

So

$$\int_0^\infty t^n e^{-\lambda t}\,dt = (0-0-\cdots 0-\cdots-0) - \left(0-0-\cdots-0-\frac{n(n-1)(n-2)\cdots 1}{\lambda^{n+1}}\right)$$

$$= \frac{n(n-1)(n-2)\cdots 1}{\lambda^{n+1}}$$

Hence,

$$1 = \int_0^\infty c\, t^n e^{-\lambda t}\, dt$$

$$= c \int_0^\infty t^n e^{-\lambda t}\, dt$$

$$= \frac{n(n-1)(n-2)\cdots 1}{\lambda^{n+1}}\, c$$

$$\Rightarrow \quad c = \frac{\lambda^{n+1}}{n(n-1)(n-2)\cdots 1} = \frac{\lambda^{n+1}}{n!}$$