Integration of Rational Functions by Partial Fractions

Monday, January 25, 2021

1. Find the partial fraction decomposition for:

$$\frac{x-29}{x^3(x^2-4)^2(x^2+x+16)^2}$$

$$\frac{x-29}{x^3(x^2-4)^2(x^2+x+16)^2} = \frac{x-29}{x^3(x-4)^2(x+4)^2(x^2+x+16)^2}$$

$$= \frac{A_1}{x} + \frac{A_2}{x^2} + \frac{A_3}{x^3} + \frac{B_1}{x-4} + \frac{B_2}{(x-4)^2} + \frac{C_1}{x+4} + \frac{C_2}{(x+4)^2} + \frac{D_1x+D_2}{x^2+x+16} + \frac{E_1x+E_2}{(x^2+x+16)^2}$$

I think the much we can learn from this problem (at this time!) is recognizing the form of partial fractions.

2. Evaluate $\int \frac{x}{1+x^4}\,dx$

Let $u = x^2$. Then $\frac{du}{dx} = 2x \Rightarrow \frac{du}{2x} = dx$

So

$$\int \frac{x}{1+x^4}dx = \int \frac{x}{1+u^2} \cdot \frac{du}{2x}$$

$$= \frac{1}{2}\int \frac{1}{1+u^2}du$$

$$= \frac{1}{2}\tan^{-1}u + C$$

$$= \frac{1}{2}\tan^{-1}(x^2) + C$$

$=$

3. Evaluate $\int \frac{r^2}{r+4}\,dr$ (the fast way)

$$\int \frac{r^2}{r+4}\,dr = \int \left(r-4 + \frac{16}{r+4}\right) dr$$

$$= \frac{r^2}{2} - 4r + 16\ln|r+4| + C$$

$$\begin{array}{r|l} & r-4 \\ \hline r+4 & r^2 \\ & r^2+4r \\ \hline & -4r \\ & -4r-16 \\ \hline & 16 \end{array}$$

4. Evaluate $\int \frac{10}{x^3-x^2+9x-9}\,dx$

$$\begin{array}{r|l} & x^2+9 \\ \hline x-1 & x^3-x^2+9x-9 \\ & x^3-x^2 \\ \hline & 9x-9 \\ & 9x-9 \\ \hline \end{array}$$

$$\frac{10}{x^3-x^2+9x-9} = \frac{10}{(x-1)(x^2+9)}$$

$$= \frac{A}{x-1} + \frac{Bx+C}{x^2+9}$$

By cover up rule,

$$A = \left.\frac{10}{x^2+9}\right|_{x=1} = \frac{10}{1^2+9} = \frac{10}{10} = 1$$

$$\frac{10}{} = \frac{1}{x-1} + \frac{Bx+C}{x^2+9}$$

$$\frac{10}{x^3-x^2+9x-9} = \frac{1}{x-1} + \frac{Bx+C}{x^2+9}$$

Plug $x=0$:

$$\frac{10}{0^3-0^2+9(0)-9} = \frac{1}{0-1} + \frac{B(0)+C}{0^2+9} \Rightarrow -\frac{10}{9} = -1 + \frac{C}{9}$$

$$\Rightarrow -10 = -9 + C$$

$$\Rightarrow C = -1$$

Thus,

$$\frac{10}{x^3-x^2+9x-9} = \frac{1}{x-1} + \frac{Bx-1}{x^2+9}$$

Now, plug $x=-1$:

$$\frac{10}{(-1)^3-(-1)^2+9(-1)-9} = \frac{1}{-1-1} + \frac{B(-1)-1}{(-1)^2+9}$$

$$\Rightarrow \frac{10}{-20} = -\frac{1}{2} + \frac{-B-1}{10}$$

$$\Rightarrow -5 = -5 - (B+1) \Rightarrow B = -1$$

ie;

$$\frac{10}{x^3-x^2+9x-9} = \frac{1}{x-1} - \frac{x+1}{x^2+9}$$

So

$$\int \frac{10}{x^3-x^2+9x-9}\,dx = \int \left(\frac{1}{x-1} - \frac{x+1}{x^2+9}\right)dx$$

$$= \int \frac{1}{x-1}\,dx - \int \frac{x}{x^2+9}\,dx - \int \frac{1}{x^2+9}\,dx$$

$$= \int \frac{1}{x-1}\,dx - \int \frac{x}{x^2+9}\,dx - \int \frac{1}{x^2+9}\,dx$$

$$= \ln|x-1| - \frac{1}{2}\ln(x^2+9) - \frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right) + C$$

5. Evaluate $\int_0^1 \frac{2x^3+5}{x^4+5x^2+4}\,dx$

u^2+5u+4
$= (u+1)(u+4)$
$= (x^2+1)(x^2+4)$

$$\frac{2x^3+5}{x^4+5x^2+4} = \frac{2x^3+5x-5x+5}{x^4+5x^2+4}$$

$$= \frac{2x^3+5x}{x^4+5x^2+4} + \frac{5-5x}{x^4+5x^2+4}$$

But

$$\frac{5-5x}{x^4+5x^2+4} = \frac{5-5x}{(x^2+1)(x^2+4)} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{x^2+4}$$

Plug $x=0$:

$$\frac{5-5(0)}{(0^2+1)(0^2+4)} = \frac{A(0)+B}{0^2+1} + \frac{C(0)+D}{0^2+4}$$

$$\Rightarrow \quad \frac{5}{4} = B + \frac{D}{4} \Rightarrow 5 = 4B + D \quad \text{—— } 1$$

Plug $x=1$:

$$\frac{5-5(1)}{(1^2+1)(1^2+4)} = \frac{A(1)+B}{1^2+1} + \frac{C(1)+D}{1^2+4}$$

$$\frac{A+B}{} + \frac{C+D}{} \Rightarrow 0 = 5A+5B+2C+2D \quad \text{—— } 2$$

$$\Rightarrow \quad 0 = \frac{A+B}{2} + \frac{C+D}{5} \Rightarrow 0 = 5A+5B+2C+2D \quad \text{—— (2)}$$

Plug $x=-1$

$$\frac{5-5(-1)}{((-1)^2+1)((-1)^2+4)} = \frac{A(-1)+B}{(-1)^2+1} + \frac{C(-1)+D}{(-1)^2+4}$$

$$\Rightarrow \quad \frac{10}{10} = \frac{-A+B}{2} + \frac{-C+D}{5}$$

$$\Rightarrow \quad 10 = -5A+5B-2C+2D \quad \text{—— (3)}$$

Plug in $x=2$:

$$\frac{5-5(2)}{(2^2+1)(2^2+4)} = \frac{A(2)+B}{2^2+1} + \frac{C(2)+D}{2^2+4}$$

$$\Rightarrow \quad \frac{-5}{40} = \frac{2A+B}{5} + \frac{2C+D}{8}$$

$$\Rightarrow \quad -5 = 16A+8B+10C+5D \quad \text{—— (4)}$$

Adding (2) and (3) gives:

$$10 = 10B + 4D \quad \text{—— (5)}$$

But (1) x4:

$$20 = 16B + 4D$$

$$-10 = -6B$$

$$\Rightarrow \quad B = \frac{10}{6} = \frac{5}{3}$$

So from (1),

$$5 = 4\left(\frac{5}{3}\right) + D \Rightarrow 5 = \frac{20}{3} + D \Rightarrow D = 5 - \frac{20}{3} = \frac{-5}{3}$$

Now, substitute $B = \frac{5}{3}$ and $D = \frac{-5}{3}$ into 2 and 4:

$$0 = 5A + 5\left(\frac{5}{3}\right) + 2C + 2\left(\frac{-5}{3}\right)$$

$$\Rightarrow \quad -5 = 5A + 2C \quad \text{———} \quad 6$$

and

$$-5 = 16A + 8\left(\frac{5}{3}\right) + 10C + 5\left(\frac{-5}{3}\right)$$

$$\Rightarrow \quad -5 = 8A + 5C \quad \text{———} \quad 7$$

Solving 6 and 7, we get

$$A = -\frac{5}{3} \quad \text{and} \quad C = \frac{5}{3}$$

Thus,

$$\frac{5-5x}{x^4+5x^2+4} = \frac{5-5x}{(x^2+1)(x^2+4)} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{x^2+4}$$

$$= \frac{-\frac{5}{3}x + \frac{5}{3}}{x^2+1} + \frac{\frac{5}{3}x - \frac{5}{3}}{x^2+4}$$

and

$$\int_0^1 \frac{2x^3+5}{x^4+5x^2+4}\,dx = \int_0^1 \frac{2x^3+5x}{x^4+5x^2+4}\,dx + \int_0^1 \frac{5-5x}{x^4+5x^2+4}\,dx$$

$$= \frac{1}{2}\int_0^1 \frac{2(2x^3+5x)}{x^4+5x^2+4}\,dx + \int_0^1 \frac{5-5x}{(x^2+1)(x^2+4)}\,dx$$

$$= \frac{1}{2}\int_0^1 \frac{4x^3+10x}{x^4+5x^2+4}\,dx + \int_0^1 \left(\frac{-\frac{5}{3}x + \frac{5}{3}}{x^2+1} + \frac{\frac{5}{3}x - \frac{5}{3}}{x^2+4} \right) dx$$

$$= \frac{1}{2}\ln\left(x^4+5x^2+4\right)\Big|_0^1 + \int_0^1 \left(\frac{-\frac{5}{3}x}{x^2+1} + \frac{\frac{5}{3}}{x^2+1} + \frac{\frac{5}{3}x}{x^2+4} - \frac{\frac{5}{3}}{x^2+4} \right) dx$$

$$= \frac{1}{2}\left(\ln(10) - \ln(4)\right) - \frac{5}{3}\int_0^1 \frac{x}{x^2+1}\,dx + \frac{5}{3}\int_0^1 \frac{1}{x^2+1}\,dx + \frac{5}{3}\int_0^1 \frac{x}{x^2+4}\,dx - \frac{5}{3}\int_0^1 \frac{1}{x^2+4}\,dx$$

$$= \frac{1}{2}\ln\left(\frac{10}{4}\right) - \frac{5}{6}\int_0^1 \frac{2x}{x^2+1}\,dx + \frac{5}{3}\int_0^1 \frac{1}{x^2+1}\,dx + \frac{5}{6}\int_0^1 \frac{2x}{x^2+4}\,dx - \frac{5}{3}\int_0^1 \frac{1}{x^2+4}\,dx$$

$$= \frac{1}{2}\ln\left(\frac{5}{2}\right) - \frac{5}{6}\ln\left(x^2+1\right) + \frac{5}{3}\tan^{-1}x + \frac{5}{6}\ln\left(x^2+4\right) - \frac{5}{3(2)}\tan^{-1}\left(\frac{x}{2}\right)\Big|_0^1$$

$$= \frac{1}{2}\ln\left(\frac{5}{2}\right) + \frac{5}{6}\ln\left(\frac{x^2+4}{x^2+1}\right) + \frac{5}{3}\tan^{-1}x - \frac{5}{6}\tan^{-1}\left(\frac{x}{2}\right)\Big|_0^1$$

$$= \frac{1}{2}\ln\left(\frac{5}{2}\right) + \frac{5}{6}\left(\ln\left(\frac{1^2+4}{1^2+1}\right) - \ln\left(\frac{0^2+4}{0^2+1}\right)\right) + \frac{5}{3}\left(\tan^{-1}1 - \tan^{-1}0\right) - \frac{5}{6}\left(\tan^{-1}\left(\frac{1}{2}\right) - \tan^{-1}0\right)$$

$$= \frac{1}{2}\ln\left(\frac{5}{2}\right) + \frac{5}{6}\left(\ln\left(\frac{5}{2}\right) - \ln(4)\right) + \frac{5}{3}\left(\frac{\pi}{4} - 0\right) - \frac{5}{6}\left(\tan^{-1}\left(\frac{1}{2}\right) - 0\right)$$

$$= \frac{1}{2}\ln\left(\frac{5}{2}\right) + \frac{5}{6}\ln\left(\frac{5}{8}\right) + \frac{5\pi}{12} - \frac{5}{6}\tan^{-1}\left(\frac{1}{2}\right)$$

6. Evaluate $\int \frac{\cos x}{(1+\cos x)(1-\cos x)+\sin x}\, dx$

$$\int \frac{\cos x}{(1+\cos x)(1-\cos x)+\sin x}\, dx = \int \frac{\cos x}{1-\cos^2 x+\sin x}\, dx$$

$$= \int \frac{\cos x}{\sin^2 x + \sin x}\, dx$$

Let $u = \sin x$. Then $\frac{du}{dx} = \cos x \Rightarrow \frac{du}{\cos x} = dx$

$$= \int \frac{\cos x}{u^2+u} \cdot \frac{du}{\cos x}$$

$$= \int \frac{1}{u(u+1)}\, du$$

$$= \int \left(\frac{1}{u} - \frac{1}{u+1}\right) du$$

$$= \ln|u| - \ln|u+1| + C$$

$$= \ln|\sin x| - \ln|\sin x + 1| + C$$

7. Let $k > 0$ and $0 < c < 1$ be constants. A rumour gets spread in town. The amount of time it takes until a fraction p of the town's population has heard the rumour is given by

$$t(p) = \int_c^p \frac{k}{x(1-x)}\, dx.$$

(a) Evaluate the integral to find a formula for $t(p)$, in terms of a single log expression.

(b) At time $t = 0$, one percent of the population has heard the rumour, and by time $t = 1$, half the population has heard the rumour. What are the values of c and k?

ⓐ $t(p) = \int_c^p \frac{k}{x(1-x)}\, dx = \int_c^p \left(\frac{k}{x} + \frac{k}{1-x}\right) dx$

$$\left(\int^p \cdots\, dx + \int^p \frac{1}{\cdots}\, dx\right)$$

$$= k\left(\int_c^p \frac{1}{x}\,dx + \int_c^p \frac{1}{1-x}\,dx\right)$$

$$= k\left(\int_c^p \frac{1}{x}\,dx - \int_c^p \frac{-1}{1-x}\,dx\right)$$

$$= k\left(\ln x - \ln(1-x)\right)\Big|_c^p$$

$$= k\ln\left(\frac{x}{1-x}\right)\Big|_c^p$$

$$= k\left(\ln\left(\frac{p}{1-p}\right) - \ln\left(\frac{c}{1-c}\right)\right)$$

$$= k\ln\left(\frac{p}{1-p} \div \frac{c}{1-c}\right)$$

$$= k\ln\left(\frac{p}{1-p} \times \frac{1-c}{c}\right)$$

$$\therefore \quad t(p) = k\ln\left(\frac{p(1-c)}{c(1-p)}\right)$$

b) $t(0.01) = 0 \Rightarrow 0 = k\ln\left(\frac{0.01(1-c)}{c(1-0.01)}\right)$

$$\Rightarrow \ln\left(\frac{0.01(1-c)}{c(0.99)}\right) = 0 \quad \text{if } k \neq 0$$

$$\Rightarrow \frac{0.01(1-c)}{c(0.99)} = 1$$

$$\Rightarrow \quad \frac{0.01(1-c)}{c(0.99)} = 1$$

$$\Rightarrow \quad 0.01(1-c) = 0.99c$$

$$\Rightarrow \quad 0.01 - 0.01c = 0.99c$$

$$\Rightarrow \quad c = 0.01$$

Also,

$t(0.5) = 1 \Rightarrow$

$$1 = k\ln\left(\frac{0.5(1-c)}{c(1-0.5)}\right)$$

$$= k\ln\left(\frac{0.5(1-0.01)}{0.01(1-0.5)}\right)$$

$$= k\ln\left(\frac{0.495}{0.005}\right)$$

$$= k\ln(99)$$

$$\Rightarrow \quad k = \frac{1}{\ln(99)}$$