Exercises 7.1: #15, 20, 29, 33, 35, 51, 52

Friday, January 15, 2021 1:01 AM

15 $\int (\ln x)^2 dx$

Let $u = \ln x$ and $dv = 1dx$. Then $v = x$ and $\frac{du}{dx} = \frac{1}{x} \Rightarrow du = \frac{1}{x}dx$

Thus,

$$\int (\ln x)^2 dx = uv - \int v\,du$$
$$= x\ln x - \int x \cdot \frac{1}{x}dx$$
$$= x\ln x - \int 1\,dx$$
$$= x\ln x - x + C$$
$$= x(\ln x - 1) + C$$

20 $\int x\tan^2 x\,dx = \int x(\sec^2 x - 1)dx$

Recall: $\sec^2 x = 1 + \tan^2 x$

$$= \int x\sec^2 x\,dx - \int x\,dx$$
$$= \int x\sec^2 x\,dx - \frac{x^2}{2}$$

Let $u = x$ and $dv = \sec^2 x\,dx$. Then $du = dx$ and $v = \tan x$

Thus,

$$\int x\tan^2 x\,dx = \int x\sec^2 x\,dx - \frac{x^2}{2}$$

$$= \left(uv - \int v\,du\right) - \frac{x^2}{2}$$

$$= x\tan x - \int \tan x\,dx$$

$$= x\tan x + \int \frac{-\sin x}{\cos x}\,dx$$

$$= x\tan x + \ln|\cos x| + C$$

(29) $\int_0^{\pi} x\sin x\cos x\,dx$

Recall $\sin(2x) = 2\sin x\cos x \Rightarrow \sin x\cos x = \frac{1}{2}\sin x\cos x$

So,

$$\int_0^{\pi} x\sin x\cos x\,dx = \frac{1}{2}\int_0^{\pi} x\sin(2x)\,dx$$

u	dv
x	$\sin(2x)$
1	$-\frac{\cos(2x)}{2}$

$$= -\frac{x\cos(2x)}{4}\Bigg|_0^{\pi} - \frac{1}{4}\int_0^{\pi}\cos(2x)\,dx$$

$$= \frac{-\pi\cos(2\pi)}{4} + \frac{0\cdot\cos(2\cdot 0)}{4} - \frac{1}{4}\cdot\frac{\sin(2x)}{2}\Bigg|_0^{\pi}$$

$$= \frac{-\pi}{4} - \frac{1}{8}\left(\sin\pi - \sin 0\right)$$

$$= -\frac{\pi}{4}$$

(33) $\int_0^{\pi/3} \sin x \ln(\cos x)\,dx$

Let $u = \ln(\cos x)$ and $dv = \sin x\,dx$. Then $v = -\cos x$ and

$$\frac{du}{dx} = \frac{-\sin x}{\cos x} \Rightarrow du = -\frac{\sin x}{\cos x}\,dx$$

Thus,

$$\int_0^{\pi/3} \sin x \ln(\cos x)\,dx = uv - \int_0^{\pi/3} v\,du$$

$$= -\ln(\cos x)\cos x\Big|_0^{\pi/3} - \int_0^{\pi/3} -\cos x \cdot -\frac{\sin x}{\cos x}\,dx$$

$$= -\ln(\cos \tfrac{\pi}{3})\cos \tfrac{\pi}{3} + \ln(\cos 0)\cos 0 - \int_0^{\pi/3} \sin x\,dx$$

$$= -\frac{\ln(\frac{1}{2})}{2} + \ln(1)\cdot 1 + \cos x\Big|_0^{\pi/3}$$

$$= \frac{1}{2}\ln 2 + \cos \tfrac{\pi}{3} - \cos 0$$

$$= \frac{1}{2}\ln 2 + \frac{1}{2} - 1$$

$$= \frac{1}{2}(\ln 2 - 1)$$

(35) $\int_1^2 x^4 (\ln x)^2\,dx$

Let $u = (\ln x)^2$ and $dv = x^4\,dx$. Then $v = \frac{x^5}{5}$ and

$$\frac{du}{dx} = 2\ln x \cdot \frac{1}{x} \Rightarrow du = \frac{2\ln x}{x}\,dx.$$

Thus,

$$\int_1^2 x^4(\ln x)^2\,dx = uv - \int_1^2 v\,du$$

$$= \frac{x^5}{5}(\ln x)^2\Big|_1^2 - \int_1^2 \frac{x^5}{5}\cdot\frac{2\ln x}{x}\,dx$$

$$= \frac{2^5}{5}(\ln 2)^2 - \frac{1^5}{5}(\ln 1) - \frac{2}{5}\int_1^2 x^4\ln x\,dx$$

$$= \frac{32}{5}(\ln 2)^2 - \frac{2}{5}\int_1^2 x^4\ln x\,dx$$

Let $u = \ln x$ and $dv = x^4dx$. Then $v = \frac{x^5}{5}$ and

$$\frac{du}{dx} = \frac{1}{x} \Rightarrow du = \frac{1}{x}dx.$$

So,

$$\int_1^2 x^4(\ln x)^2dx = \frac{32}{5}(\ln 2)^2 - \frac{2}{5}\int_1^2 x^4\ln x\,dx$$

$$= \frac{32}{5}(\ln 2)^2 - \frac{2}{5}\left(\frac{x^5}{5}\ln x\Big|_1^2 - \frac{1}{5}\int_1^2 x^5\cdot\frac{1}{x}dx\right)$$

$$= \frac{32}{5}(\ln 2)^2 - \frac{2}{5}\left(\frac{2^5}{5}\ln 2 - \frac{1}{5}\ln 1 - \frac{1}{5}\int_1^2 x^4dx\right)$$

$$= \frac{32}{5}(\ln 2)^2 - \frac{2}{5}\left(\frac{32}{5}\ln 2 - \frac{1}{5}\cdot\frac{x^5}{5}\Big|_1^2\right)$$

$$= \frac{32}{5}(\ln 2)^2 - \frac{64}{25}\ln 2 + \frac{2}{25}\left(\frac{2^5}{5} - \frac{1}{5}\right)$$

$$= \frac{32}{5}(\ln 2)^2 - \frac{64}{25}\ln 2 + \frac{2}{25}\left(\frac{31}{5}\right)$$

$$= \frac{32}{5}(\ln 2)^2 - \frac{64}{25}\ln 2 + \frac{62}{125}$$

51 $\int (\ln x)^n dx$

Let $u = (\ln x)^n$ and $dv = dx$. Then $v = x$ and

$$\frac{du}{dx} = \frac{n(\ln x)^{n-1}}{x} \Rightarrow du = \frac{n(\ln x)^{n-1}}{x}dx$$

Thus,

$$\int (\ln x)^n dx = uv - \int v\,du$$

$$= x(\ln x)^n - \int x \frac{n(\ln x)^{n-1}}{x} dx$$

$$= x(\ln x)^n - n\int (\ln x)^{n-1} dx$$

52 $\int x^n e^x dx$

Let $u = x^n$ and $dv = e^x dx$. Then $v = e^x$ and

$\frac{du}{dx} = nx^{n-1}$. Thus,

$$\int x^n e^x dx = uv - \int v\,du$$

$$= x^n e^x - n\int x^{n-1} e^x \, dx$$

Integration By Parts

Tuesday, January 12, 2021

1. Let f be twice differentiable with $f(0) = 6$, $f(1) = 5$, and $f'(1) = 2$. Find

$$\int_0^1 x f''(x)\, dx.$$

Let $u = x$ and $dv = f''(x)$. Then $du = dx$ and $v = f'(x)$.

So,

$$\int_0^1 x f''(x)dx = uv - \int_0^1 v\,du$$

$$= x f'(x)\Big|_0^1 - \int_0^1 f'(x)\,dx$$

$$= \left(1 f'(1) - 0 f'(0)\right) - f(x)\Big|_0^1$$

$$= 2 - \left(f(1) - f(0)\right)$$

$$= 2 - (5 - 6)$$

$$= 3$$

2. Find $\int e^x \sin(2x)\, dx$

Recall: $\boxed{\int u\,dv = uv - \int v\,du}$

Let $u = \sin(2x)$ and $dv = e^x$. Then $\frac{du}{dx} = 2\cos(2x) \Rightarrow du = 2\cos(2x)dx$

and $dv = e^x \Rightarrow v = e^x$. Thus,

$$\int e^x \sin(2x)\,dx = uv - \int v\,du$$

$$= (\sin(2x))(e^x) - 2\int e^x \cos(2x)\,dx$$

$$= e^x \sin(2x) - 2\int e^x \cos(2x)\,dx$$

Now, let $u = \cos(2x)$ and $dv = e^x$. Then $\frac{du}{dx} = -2\sin(2x)$

$\Rightarrow$ $du = -2\sin(2x)\,dx$ and $v = e^x$

Thus,

$$\int e^x \sin(2x)dx = e^x \sin(2x) - \int -2\,e^x \cos(2x)\,dx$$

$$= e^x \sin(2x) + 2\left(\cos(2x)\,e^x - 2\int e^x \sin(2x)\,dx\right)$$

$$= e^x \sin(2x) - 2e^x \cos(2x) - 4\int e^x \sin(2x)\,dx$$

$$\Rightarrow \int e^x \sin(2x)dx + 4\int e^x \sin(2x)\,dx = e^x \sin(2x) - 2e^x \cos(2x)$$

$$\Rightarrow \quad 5\int e^x \sin(2x)\,dx = e^x \sin(2x) - 2e^x \cos(2x)$$

$$\Rightarrow \quad \int e^x \sin(2x)\,dx = \frac{e^x}{5}\left(\sin(2x) - 2\cos(2x)\right) + C$$

3. Find $\int t^5 \cos(t^3)\,dt$

Here, we first reduce the complexity of the problem by performing u-substitution.

stitution.

Let $u = t^3$. Then $\frac{du}{dt} = 3t^2 \Rightarrow \frac{du}{3t^2} = dt$.

Thus,

$$\int t^5 \cos(t^3)\,dt = \int t^5 \cos u \cdot \frac{du}{3t^2}$$

$$= \frac{1}{3}\int \underbrace{t^3}_{u} \cos u \, du$$

$$= \frac{1}{3}\int u \cos u \, du$$

$$= \frac{1}{3}\left(uv - \int v\,du\right)$$

$$= \frac{1}{3}\left(u\sin u - \int \sin u\,du\right)$$

$$= \frac{1}{3}\left(u\sin u + \cos u\right)$$

$$= \frac{1}{3}\left(t^3\sin(t^3) + \cos(t^3)\right) + C$$

u	dv
u	$\cos u$
1	$\sin u$

4. Find $\int_1^4 x^{3/2} \ln x \, dx$

Let $u = \ln x$ and $dv = x^{3/2}$. Then $\frac{du}{dx} = \frac{1}{x} \Rightarrow x\,du = dx$

and $v = \frac{2}{5}x^{5/2}$. Thus,

$$\int_1^4 x^{3/2} \ln x\,dx = uv - \int_1^4 v\,du$$

and $v = \frac{2}{5} x^{5/2}$

$$\int_1^4 x^{3/2} \ln x \, dx = uv - \int_1^4 v \, du$$

$$= \frac{2}{5} x^{5/2} \ln x \Big|_1^4 - \int_1^4 \frac{2}{5} x^{5/2} \cdot \frac{1}{x} \, dx$$

$$= \frac{2}{5}\left(4^{5/2} \ln 4 - 1^{5/2} \ln 1\right) - \frac{2}{5} \int_1^4 x^{3/2} \, dx$$

$$= \frac{2}{5}\left(32 \ln 4\right) - \frac{2}{5} \cdot \frac{2}{5} x^{5/2} \Big|_1^4$$

$$= \frac{64}{5} \ln 4 - \frac{4}{25}\left(4^{5/2} - 1^{5/2}\right)$$

$$= \frac{64}{5} \ln 4 - \frac{4}{25}(32 - 1)$$

$$= \frac{320 \ln 4 - 124}{25}$$

5. Use integration by parts to verify the reduction formula:

$$\int \cos^n x \, dx = \frac{1}{n} \cos^{n-1} x \, \sin x + \frac{n-1}{n} \int \cos^{n-2} x \, dx$$

$$\int \cos^n x \, dx = \int \cos x \cos^{n-1} x \, dx.$$

Let $u = \cos^{n-1} x$ and $dv = \cos x$. Then $\frac{du}{dx} = -(n-1) \cos^{n-2} x \sin x$

$\Rightarrow du = -(n-1) \cos^{n-2} x \sin x$ and $v = \sin x$

$$\Rightarrow \quad du = -(n-1)\cos x \sin x \quad \text{and} \quad v = \sin x$$

Thus,

$$\int \cos^n x\,dx = uv - \int v\,du$$

$$= \cos^{n-1} x \sin x - \int \sin x \left(-(n-1)\cos^{n-2} x \sin x\right) dx$$

$$= \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x \sin^2 x\,dx$$

$$= \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x \left(1-\cos^2 x\right) dx$$

Recall:

$\sin^2 x + \cos^2 x = 1$

$$= \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x\,dx - (n-1)\int \cos^n x\,dx$$

$$\Rightarrow \int \cos^n x\,dx + (n-1)\int \cos^n x\,dx = \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x\,dx$$

$$\Rightarrow \quad n\int \cos^n x\,dx = \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x\,dx$$

$$\Rightarrow \quad \int \cos^n x\,dx = \frac{1}{n}\cos^{n-1} x \sin x + \frac{(n-1)}{n}\int \cos^{n-2} x\,dx$$