More on Areas and Volumes

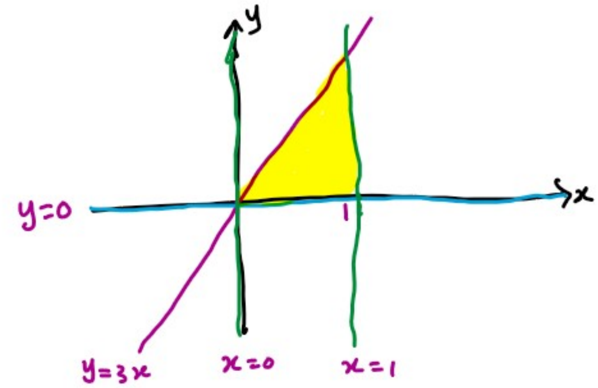
Wednesday, February 17, 2021

1. Determine whether the following definite integrals represent an area or a volume. Determine what shape is described (e.g. triangle, sphere, cone). Sketch the planar region or solid and label its dimensions.

(a) $\int_0^1 3x\,dx$

$$\int_0^1 3x\,dx = \int_{x=0}^{x=1} [y_u(x) - y_l(0)]\,dx$$ where y_u and y_l represent upper and lower curves respectively with $y_u(x) = 3x$ and $y_l(x) = 0$.

So the integral represents the area bounded by $y = 3x$, $y = 0$, $x = 0$, and $x = 1$ which is a triangle.

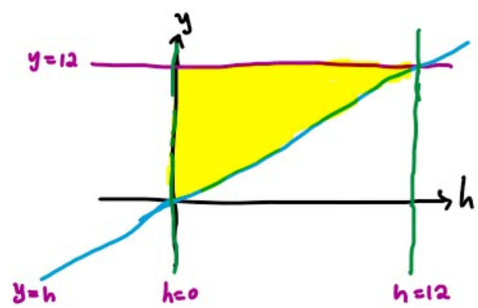


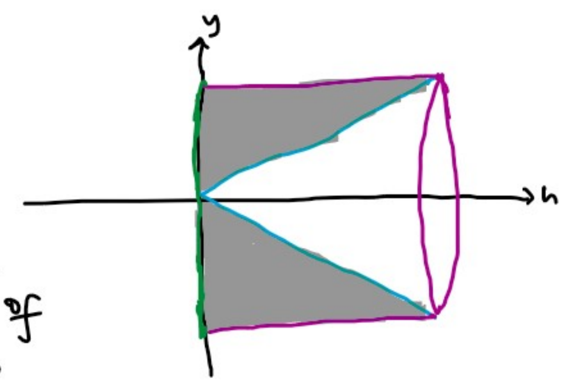
(b) $\int_0^{12} \pi(144 - h^2)\,dh$

$$\int_0^{12} \pi(144 - h^2)\,dh = \int_{h=0}^{h=12} \pi(12^2 - h^2)\,dh$$

$$= \int_0^{12} \pi(r_o^2 - r_i^2)\,dh$$

$$= \int_0^{12} \pi(y_o^2 - y_i^2)\,dh$$



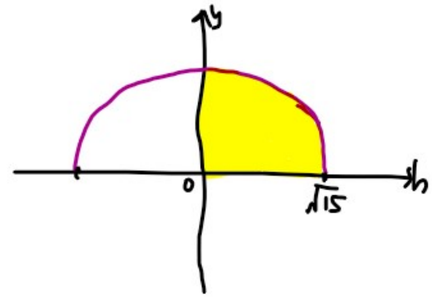


So the integral represent the volume of the solid (described in the diagram) gotten by rotating the area of the triangle bounded by $y = 12$, $y = h$, $h = 0$ and $h = 12$ about the h-axis.

(c) $\int_0^{\sqrt{15}} \sqrt{15 - h^2}\, dh$

$y = \sqrt{15-h^2}\,, y \geq 0 \iff y^2 + h^2 = 15,\ y \geq 0$

$$\int_0^{\sqrt{15}} \sqrt{15-h^2}\, dh = \int_{h=0}^{h=\sqrt{15}} \left[y_u(h) - y_l(h)\right] dh$$



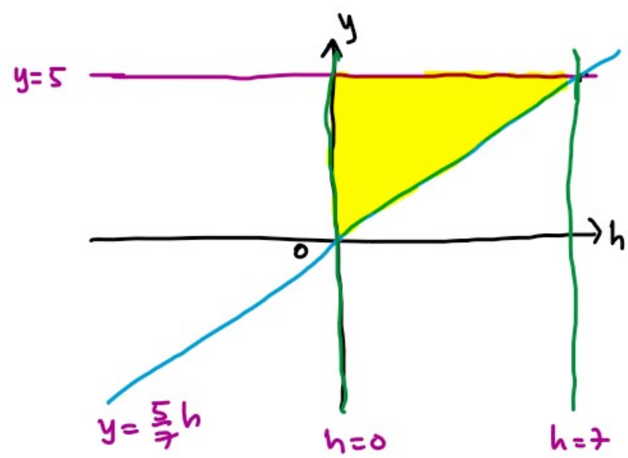
where $y_u(h) = \sqrt{15-h^2}$, $y_l(h) = 0$

So the integral represents the area of a quarter circle of radius $\sqrt{15}$ bounded by $y = \sqrt{15-h^2}$, $y=0$, $h=0$ and $h=\sqrt{15}$.

(d) $\int_0^7 5(1 - \frac{h}{7})\, dh$

$$\int_0^7 5\left(1 - \frac{h}{7}\right) dh = \int_0^7 \left(5 - \frac{5}{7}h\right) dh$$

$$= \int_{h=0}^{h=7} \left[y_u(h) - y_l(h)\right] dh$$



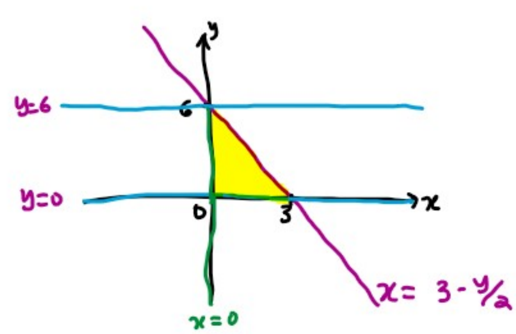
where $y_u(h) = 5$, $y_l(h) = \frac{5}{7}h$

So the integral represents the area of a triangle bounded by $y=5$, $y=\frac{5}{7}h$, $h=0$ and $h=7$.

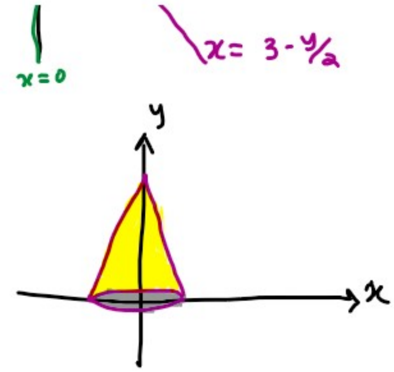
(e) $\int_0^6 \pi(3 - \frac{y}{2})^2\, dy$

$$\int_0^6 \pi\left(3 - \frac{y}{2}\right)^2 dy = \int_{y=0}^{y=6} \pi\,(r)^2 dy$$

$$= \int^6 \pi\left[x(y)\right]^2 dy$$



$$= \int_0^6 \pi [x(y)]^2 \, dy$$



where $x(y) = 3 - \frac{y}{2}$.

So the integral represents a conic solid formed by rotating the triangle bounded by $x = 3 - y/2$, $x=0$, $y=0$ and $y=6$ about the y-axis.

2. Consider the region in the first quadrant bounded by the curves $y = \sqrt{x}$ and $y = \frac{x^2}{8}$.

   (a) Find the volume of the solid with this region as its base and cross-section perpendicular to the x-axis that are squares.

   (b) Find the volume of the solid with this region as its base and cross-section perpendicular to the y-axis that are squares.

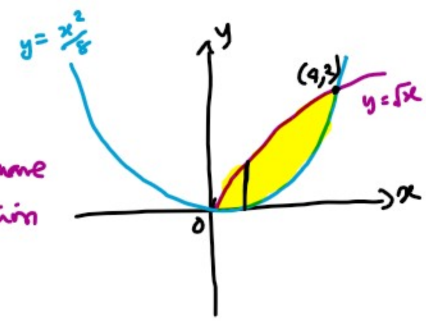
   (c) What is the volume of the solid obtained by revolving this region about the y-axis?

$\sqrt{x} = y = \frac{x^2}{8}$

$\Rightarrow 64x - x^4 = 0$

$\Rightarrow x(64 - x^3) = 0$

$\Rightarrow x = 0, x = 4$



a

(a) Cross-section perpendicular to the x-axis

So, Volume $= \int_0^4 A(x)\,dx$ and $A(x) = s^2$ where s is a side of the square of a cross-section

$= [y_u(x) - y_l(x)]^2$

$= \left[\sqrt{x} - \frac{x^2}{8}\right]^2$

$= x - \frac{1}{4}x^{5/2} + \frac{x^4}{64}$

$$= \int_0^4 \left(x - \frac{1}{4}x^{5/2} + \frac{x^4}{64}\right) dx$$

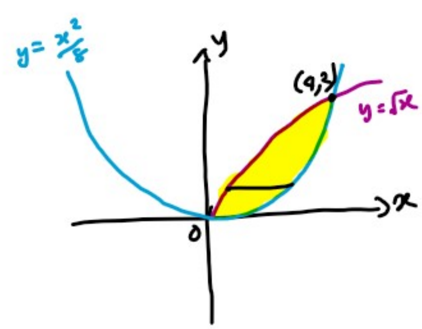
$$= \frac{x^2}{2} - \frac{\;}{9} x^{7/2} + \frac{x^5}{320} \Big|_0^4$$

$$= \frac{4^2}{2} - \frac{1}{\;}\left(4^{7/2}\right) + \frac{4^5}{320}$$

$$= 8 - \frac{1}{14}(128) + \frac{1024}{320}$$

$$= \frac{17920 - 20480 + 7168}{2240}$$

$$= \frac{4608}{2240}$$



b

(b) Volume $= \int_0^2 A(y)\,dy$ and $A(y) = s^2$

$= \left[x_r(y) - x_l(y)\right]^2$

$= \left[y^2 - \sqrt{8y}\right]^2$

$= y^4 - \sqrt{8}\, y^{5/2} + 8y$

$$= \int_0^2 \left(y^4 - \sqrt{8}\, y^{5/2} + 8y\right) dy$$

$$= \frac{y^5}{5} - \frac{2\sqrt{8}}{7} y^{7/2} + 4y^2 \Big|_0^2$$

$$= \frac{2^5}{5} - \frac{2\sqrt{8}}{7}\left(2^{7/2}\right) + 4\left(2^2\right)$$

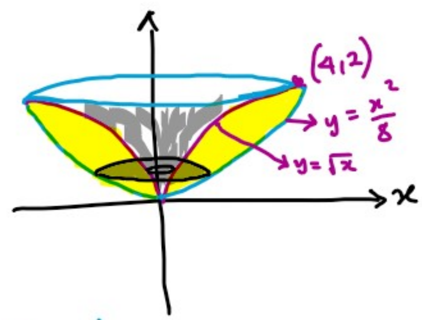
$$= \frac{32}{5} + 16 - \frac{2\cdot 2\sqrt{2}}{7}\left(2^{7/2}\right)$$

$$= \frac{112}{5} - \frac{4}{7}\left(2^4\right)$$

$$= \frac{112}{5} - \frac{64}{7}$$

$$= \frac{784 - 320}{35}$$

$$= \frac{464}{35}$$



(c) Since the region is revolved around y-axis,

Volume $= \int_0^2 A(y)\,dy$

But $A(y) = \pi\left(r_o^2 - r_i^2\right)$

$= \pi\left(x_o^2 - x_i^2\right)$

$= \pi\left[\left(\sqrt{8y}\right)^2 - \left(y^2\right)^2\right]$

$= \pi\left(8y - y^4\right)$

$$= \int_0^2 \pi\left(8y - y^4\right) dy$$

$$= \pi\left(4y^2 - \frac{y^5}{5}\right)\Big|_0^2$$

$$= \pi\left(4(2^2) - \frac{2^5}{5}\right)$$

$$= \pi\left(16 - \frac{32}{5}\right)$$

$$= \pi \left(16 - \frac{32}{5}\right)$$

$$= \frac{48\pi}{5}$$

This assumption is necessary for the problem to make sense and was omitted in your printed worksheet

3. Challenging problem: A bowl is shaped like a hemisphere with radius 30 cm. A heavy ball of diameter 10 cm is placed in the bottom of the bowl and water is poured in the bowl to a height of $0 < h < 15$ cm. Find the volume of water in the bowl.

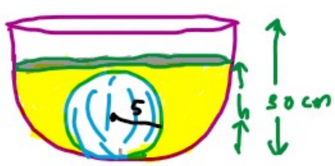
Based on the information given, the ball will always settle at the bottom of the ball and so there are two possibilities:

(a) the ball is submerged in the water. ie, $10 \le h < 15$

(b) part of the ball in the water while other parts lie above $(0 < h < 10)$

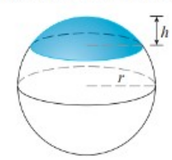
We now consider these cases one-by-one:

(a) The ball is submerged in the water $(10 \le h < 15)$



Compare with this problem from 6.2:

49. A cap of a sphere with radius r and height h



which had

$$\text{Volume of cap} = \frac{\pi}{3}\left(3rh^2 - h^3\right)$$

So,

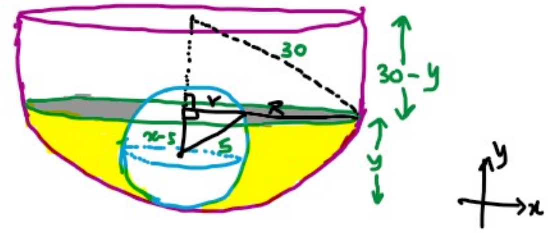
$$\text{Volume of water in the bowl} = \text{“Volume of cap”} - \text{volume of ball}$$

$$= \frac{\pi}{3}\left(3(30)h^2 - h^3\right) - \frac{4}{3}\pi(5)^3$$

$$= \frac{\pi}{3}\left(90h^2 - h^3 - 500\right) \text{ cm}^3$$

(b) Part of the ball in the water while other parts lie above $(0 < h < 10)$

Assume the water level is as shown and consider two disks formed on the bowl and on the ball at any point y where $0 < y < h$.



Let R and r be radii of these disks respectively.

Then, area of washer

$$A(y) = \pi(R^2 - r^2).$$

But by Pythagoras rule,

$$R^2 + (30-y)^2 = 30^2 \quad \text{and} \quad r^2 + (y-5)^2 = 5^2$$

$$\Rightarrow R^2 = 30^2 - (30-y)^2 \quad \text{and} \quad r^2 = 5^2 - (y-5)^2$$

$$= y(60-y) \qquad\qquad = (10-y)y$$

$$= 60y - y^2 \qquad\qquad = 10y - y^2$$

$$= \pi\left[(60y - y^2) - (10y - y^2)\right]$$

$$= \pi\left[60y - y^2 - 10y + y^2\right]$$

$$= \pi(60y - 10y)$$

$$= 50\pi y$$

Hence,

$$\text{Volume of water} = \int_0^h A(y)\,dy$$

$$= \int_0^h 50\pi y\,dy$$

$$= 25\pi y^2 \Big|_0^h$$

$$= \underline{\underline{25h^2 \text{ cm}^3}}$$

4. Example 9 on §6.2: A wedge is cut out of a circular cylinder of radius 4 by two planes. One plane is perpendicular to the axis of the cylinder. The other intersects the first at an angle of 30° along a diameter of the cylinder. Find the volume of the wedge.

Solution: Try to understand the solution given in the book.

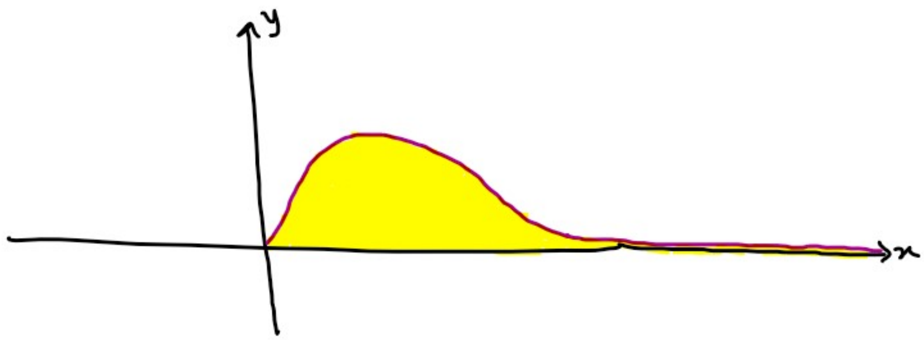
5. A couple of weeks ago we derived the "normalization constant" for Gamma probability density functions through the following indefinite integral:

$$\int_0^\infty t^n e^{-\lambda t}\,dt = \frac{n!}{\lambda^{n+1}},$$

for a positive integer n (the "shape parameter") and constant $\lambda > 0$ (the "rate parameter").

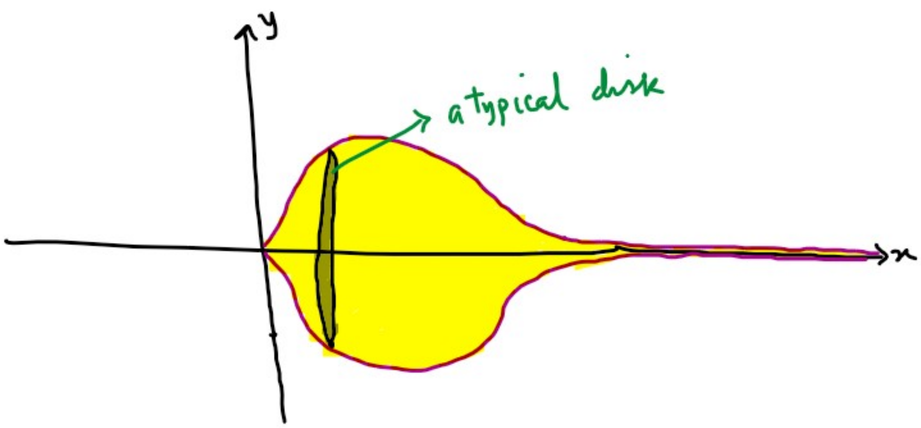
Use this result to find the volume of the solid generated by revolving the region in the first quadrant below the curve $y = x^n e^{-\lambda x}$ about the x-axis.

The graph of $y = x^n e^{-\lambda x}$ for each n and each λ looks like



and the corresponding region it makes with the x-axis is shaded.

The solid generated will look like:



with

Volume $= \int^{\infty} A(x)\,dx$

But $A(x) = \pi r^2$

$= \pi(x^n e^{-\lambda x})^2$

with

$$\text{Volume} = \int_0^\infty A(x)dx$$

$$= \int_0^\infty \pi x^{2n} e^{-2\lambda x} dx$$

$$= \pi \int_0^\infty x^{2n} e^{-2\lambda x} dx$$

$$= \pi \left(\frac{(2n)!}{(2\lambda)^{2n+1}} \right)$$

$$= \frac{(2n)!\,\pi}{(2\lambda)^{2n+1}}$$

But $A(x) = \pi r^2$

$$= \pi \left(x^n e^{-\lambda x}\right)^2$$

$$= \pi x^{2n} e^{-2\lambda x}$$

using the fact:

$$\int_0^\infty t^n e^{-\lambda t} dt = \frac{n!}{\lambda^{n+1}}$$

with n as $2n$ and λ as 2λ