Questions for recitation 7 April 2021

1. What is wrong with the following computation (for $x > 0$?)

$$2 < (1 + x + x^2 + x^3 + \cdots + x^n + \dots) + (1 + \frac{1}{x} + \frac{1}{x^2} + \frac{1}{x^3} + \cdots + \frac{1}{x^n} + \dots)$$

$$= \frac{1}{1-x} + \frac{1}{1-\frac{1}{x}} = \frac{1}{1-x} - \frac{x}{1-x} = 1$$

Solution: The use of $\frac{a}{1-r}$ requires $|r| < 1$. The second line attempts to simultaneously simplify geometric series with $r = x$ and $r = \frac{1}{x}$, and at most one of these can satisfy $|r| < 1$. Note that if we consider e.g. $x = 2$, using $\frac{1}{1-x}$ implies that $-1 = \sum_{n=0}^{\infty} 2^n$, which is clearly wrong.

2. For what values of x do the following series converge (i) absolutely, (ii) conditionally? Justify.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n^5(x-3)^{2n}}{4^n}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{n\left[\ln(x)\right]^{n-1}}{x}$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n x^n}{(3n)!n!}$

Solution:

a) Taking the ratio test:

$$\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = |\frac{(n+1)^5}{n^5} \cdot \frac{(x-3)^{2n+2}}{(x-3)^{2n}} \cdot \frac{4^n}{4^{n+1}}| = \frac{(x-3^2)}{4}$$

We then have absolute convergence when $\frac{(x-3)^2}{4} < 1$, or $-4 < (x-3)^2 < 4$, so $1 < x < 5$. This is the interior of our interval, but we should also check the endpoints. When $x = 1$, our sum reduces to $\sum \frac{n^5 2^{2n}}{4^n} = \sum n^5$, which diverges by the test for divergence. We get an identical sum when $x = 5$, so the endpoints do not converge, and $(1, 5)$ is our interval of absolute convergence, with other x diverging.

b) This is not a power series, so we may get a strange looking interval. We may lose symmetry and not be able to characterize the interval as having a "radius". Also note that we have to exclude $x \leq 0$ for domain issues. We can still use the ratio test, however, and: $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = \ln(x)$, so we get absolute convergence when $|\ln(x)| < 1$, so $e^{-1} < x < e$. At the left endpoint our sum is $\sum n(-1)^{n-1} \cdot e$, which diverges by the test for divergence. The endpoint for $x = e$ is identical without the alternation (also divergent), so our interval for convergence is $e^{-1} < x < e$, over which the convergence is absolute.

c) By the ratio test:

$$\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = \lim_{n\to\infty} \frac{(2x)^{n+1}}{(2x)^n} \cdot \frac{(3n)!}{(3n+3)!} \cdot \frac{n!}{(n+1)!} = \lim_{n\to\infty} 2x \cdot \frac{1}{(3n+3)(3n+2)(3n+1)} \frac{1}{n+1} \to 0$$

for all real x. So this sum converges absolutely for an interval of $x \in (-\infty, \infty)$.

3. For the following power series, find the interval of convergence and the sum of the series where it is convergent.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(x-1)^{2n}}{4^n}$

(b) $\displaystyle\sum_{n=0}^{\infty} (x-1)^n$

Solution:

a) Rewriting reveals that this is geometric: $\displaystyle\sum_{n=1}^{\infty} \frac{(x-1)^{2n}}{4^n} = \sum_{n=1}^{\infty} \left(\frac{(x-1)^2}{4}\right)^n$, so the I.O.C. is $|\frac{(x-1)^2}{4}| < 1 \implies -1 < x < 3$, where the series diverges at the endpoints and converges absolutely to $\frac{(x-1)^2/4}{1-(x-1)^2/4}$ inside the interval.

b) Again, the series is geometric, so it converges absolutely when $|x-1| < 1 \implies 0 < x < 2$ and this convergence is absolute on the interior (to $\frac{1}{2-x}$) and divergent for other x.

4. Find the radius of convergence and the interval of convergence for the following series:

(a) $\displaystyle\sum_{n=1}^{\infty} \left(1 + \frac{2}{n}\right)^n x^n$

(b) $\displaystyle\sum_{n=1}^{\infty} n!(x-\pi)^n$

(c) $\displaystyle\sum_{n=3}^{\infty} \left(\frac{x^2-1}{2}\right)^n$ (Note: this does not match our usual form of a "power series"; it is not centered properly. For this reason, the notion of "radius of convergence" may not be well defined as the series may lack symmetry).

Solution:

a) Root test: $\displaystyle\lim_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty} (1 + \frac{2}{n})x = x$. So this converges absolutely when $|x| < 1$ and diverges when $|x| > 1$. At the endpoints, we have $\displaystyle\sum \left(1 + \frac{2}{n}\right)^n$ and the same sum, but alternating. However, $\lim_{n\to\infty} \left(1 + \frac{2}{n}\right)^n = e^2$, so this series diverges by the test for divergence. Since the endpoints diverge, our interval of convergence is $(-1, 1)$. (Then $R = 1$.)

b) Ratio test: $\displaystyle\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = \lim_{n\to\infty} (n+1)(x-\pi)$. This diverges unless $x = \pi$, where the series converges absolutely, as each term would be zero. (Then $R = 0$.)

c) By the root test: $\lim_{n\to\infty} |a_n|^{1/n} = |\frac{x^2-1}{2}|$. This is less than 1 if $-2 < x^2 - 1 < 2$, which holds when $-1 < x^2 < 3$ or $|x| < \sqrt{3}$. At both ends the series simplifies to $\sum 1^n$ (divergent), so convergence is limited to only the inside of the interval $(-\sqrt{3}, \sqrt{3})$.