Questions for recitation 9 April 2021

1. Consider the power series below. For what values of x does the series converge absolutely? Conditionally? What is the interval of convergence?

$$\sum_{n=1}^{\infty} \frac{(-1)^n (x-4)^n}{n 2^n}$$

Solution: By the ratio test: $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = \frac{|x-4|}{2}$, where this is less than one for $2 < x < 6$. For $x = 2$, this simplifies to the harmonic series and diverges. For $x = 6$ this simplifies to the alternating harmonic series and converges conditionally. Thus, the interval of convergence is $2 < x \leq 6$ where the interior of the interval is absolute convergence and $x = 6$ is conditional.

2. Consider $g(x) = \ln(1 + x)$.

   (a) Using $\frac{1}{1+t} = \sum_{n=0}^{\infty} (-t)^n$ for $|t| < 1$, find a series for $g(x)$ and the associated radius of convergence.

   (b) Suppose we wish to evaluate $\int_0^1 xg(x)\,dx$ via series. How many terms of the associated series would we need to use to ensure that our result is within .01 of the correct value?

Solution:

a) $\frac{d}{dx} g(x) = \frac{1}{1+x} \overset{|x|<1}{=} \sum_{n=0}^{\infty} (-x)^n$. Then we can integrate both sides to get: $g(x) = \int \sum_{n=0}^{\infty} (-x)^n\,dx = \sum_{n=0}^{\infty} \int (-1)^n (x)^n\,dx = \sum_{n=0}^{\infty} (-1)^n C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}$. We then note that if $x = 0$, our series should also equal zero (since $\ln(1) = 0$), so the constant of integration is 0. Then $g(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}$

b) Using a), we are interested in finding $\int_0^1 xg(x)\,dx = \int_0^1 \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+2}}{n+1}\,dx = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+3}}{(n+1)(n+3)}\Big|_0^1 = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)(n+3)}$. Using the alternating series remainder theorem, we should include terms until $(n+1)(n+3) > 100$, which occurs at $n = 9$. We include 9 terms (since the index of the series begins at 0).

3. Noting that the geometric series satisfies $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ for $|x| < 1$, determine power series expansions for the following functions. Also determine the relevant radii of convergence.

   (a) $\frac{1}{1+x^2}$

   (b) $\frac{1}{x+2}$

   (c) $\frac{x^3}{x+2}$

Solution:

a) $$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} \overset{|x^2|<1}{=} \sum_{n=0}^{\infty}(-x^2)^n = \sum_{n=0}^{\infty}(-1)^n x^{2n}$$

b) $$\frac{1}{x+2} = \frac{1}{2}\cdot\frac{1}{1-(\frac{-x}{2})} \overset{|x|<2}{=} \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{-x}{2}\right)^n = \sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}x^n$$

c) Using part b): $$\frac{x^3}{x+2} = x^3\cdot\frac{1}{x+2} \overset{|x|<2}{=} x^3\cdot\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}x^n = \sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}x^{n+3}$$