Sequences

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1. Find a plausible formula for the nth term in the sequence. Note that n should start at 1 in each case.

(a) 0,1,1,2,3,5,8,13,...

This is the Fibonacci sequence with

$a_1 = 0$, $a_2 = 1$ and $a_n = a_{n-1} + a_{n-2}$ for $n \geq 3$.

$a_1 = 0$

$a_2 = 1$

$a_3 = 1 = a_2 + a_1$

$a_4 = 2 = a_3 + a_2$

$a_5 = 3 = a_4 + a_3$

$a_6 = 5 = a_5 + a_4$

$a_7 = 8 = a_6 + a_5$

$a_8 = 13 = a_7 + a_6$

$\vdots$

$a_n = a_{n-1} + a_{n-2}$

(b) 0,1,0,1,0,1,...

$a_1 = 0$

$a_2 = 1 = (-1)^2 + a_1$

$a_3 = 0 = (-1)^3 + a_2$

$a_4 = 1 = (-1)^4 + a_3$

$a_5 = 0 = (-1)^5 + a_4$

$a_6 = 1 = (-1)^6 + a_5$

$\vdots$

$a_n = (-1)^n + a_{n-1}$

Also,

$$a_n = \frac{1-(-1)^{n+1}}{2}; \quad a_n = \frac{(-1)^n + 1}{2}; \quad a_n = a_{n-2},\ a_1 = 0, a_2 = 1$$

are among many other possible representations of the sequence.

(c) 1,0,-1,0,1,0,... (Think of how trig functions change signs in different quadrants)

$a_1 = 1 = \sin(\frac{\pi}{2})$

$a_2 = 0 = \sin\pi = \sin\left(\frac{2\pi}{2}\right)$

$a_3 = -1 = \sin\left(\frac{3\pi}{2}\right)$

$a_4 = 0 = \sin(0\pi) = \sin\left(\frac{4\pi}{2}\right)$

$\vdots$

$a_n = \sin\left(\frac{n\pi}{2}\right)$

(This is the most intuitive since $\sin\theta$ is $+$ on 1^{st} and 2^{nd} quadrants but $-$ on the 3^{rd})

However, we could derive a lot more by playing around with the trig identities

$$\sin(\alpha + \beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$$

$$\sin(\alpha - \beta) = \sin(\alpha)\cos(\beta) - \cos(\alpha)\sin(\beta)$$

$$\cos(\alpha + \beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$$

$$\cos(\alpha - \beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)$$

For instance,

$$\cos\left(\frac{n-1}{2}\pi\right) = \cos\left(\frac{n\pi}{2} - \frac{\pi}{2}\right)$$

$$= \cos\left(\frac{n\pi}{2}\right)\cos\left(-\frac{\pi}{2}\right) + \sin\left(\frac{n\pi}{2}\right)\sin\left(\frac{\pi}{2}\right)$$

$$= \quad 0 \quad + \sin\left(\frac{n\pi}{2}\right)$$

$$= \sin\left(\frac{n\pi}{2}\right).$$

So $a_n = \cos\left(\frac{(n-1)\pi}{2}\right)$ is another possibility.

Even more intuitive for this problem is the representation:

$$a_1 = 1$$
$$a_2 = 0$$
$$a_3 = -1$$
$$a_4 = 0$$
$$a_n = a_{n-4}, \quad n \geq 5$$

and the idea stems from the fact that the sequence repeats itself after the 4th term (Thanks to the student who suggested this at one of the recitations)

(d) -3,-2,-1,0,1,...

$$a_1 = -3$$
$$a_2 = -2 = a_1 + 1$$
$$a_3 = -1 = a_2 + 1$$
$$a_4 = 0 = a_3 + 1$$
$$a_5 = 1 = a_4 + 1$$
$$\vdots$$
$$a_n = a_{n-1} + 1$$

2. Consider the sequence given by $a_1 = 1$, $a_n = \left(1 - \frac{1}{n^2}\right) a_{n-1}$.

(a) Write out the first 4 terms of the sequence.

$a_1 = 1, \quad a_n = \left(1 - \frac{1}{n^2}\right) a_{n-1}$

So,

$a_2 = \left(1 - \frac{1}{2^2}\right) a_{2-1} = \left(1 - \frac{1}{4}\right) a_1 = \left(1 - \frac{1}{4}\right) 1 = \frac{3}{4},$

$a_3 = \left(1 - \frac{1}{3^2}\right) a_{3-1} = \left(1 - \frac{1}{9}\right) a_2 = \left(\frac{9-1}{9}\right) \frac{3}{4} = \frac{8}{9} \cdot \frac{3}{4} = \frac{2}{3},$

$a_4 = \left(1 - \frac{1}{4^2}\right) a_{4-1} = \left(1 - \frac{1}{16}\right) a_3 = \left(\frac{16-1}{16}\right) \frac{2}{3} = \frac{15}{16} \cdot \frac{2}{3} = \frac{5}{8}.$

3. A ball is dropped from a height of ten feet and bounces. Each bounce is $\frac{3}{4}$ the height of the bounce before. So, after the ball hits the floor for the first time, the ball rises to a height of $10(\frac{3}{4}) = 7.5$ feet, and after it hits the floor for a second time, it rises to a height of $7.5(\frac{3}{4}) = 10(\frac{3}{4})^2 = 5.625$ feet.

(a) What height does the ball rise to after it hits the floor for the nth time?

(b) Find an expression for the total vertical distance the ball has travelled when it hits the ground for the first, second, and third time.

(c) Find an expression for the total vertical distance the ball has travelled when it hits the ground for the nth time.

a $\quad a_1 = 10\left(\frac{3}{4}\right)$ (1st bounce)

$a_2 = \frac{3}{4} a_1 = \frac{3}{4} \cdot 10\left(\frac{3}{4}\right) = 10\left(\frac{3}{4}\right)^2$ (2nd bounce)

$a_3 = \frac{3}{4} a_2 = \frac{3}{4} \cdot 10\left(\frac{3}{4}\right)^2 = 10\left(\frac{3}{4}\right)^3$ (3rd bounce)

$a_4 = \frac{3}{4} a_3 = \frac{3}{4} \cdot 10\left(\frac{3}{4}\right)^3 = 10\left(\frac{3}{4}\right)^4$ (4th bounce)

$$a_4 = \frac{3}{4} a_3 = \frac{3}{4} \cdot 10\left(\frac{3}{4}\right)^3 = 10\left(\frac{3}{4}\right)^4 \quad (4^{th} \text{ bounce})$$

$$\vdots$$

$$a_n = \frac{3}{4}\left(\frac{3}{4}\right)^n \quad (n^{th} \text{ bounce})$$

(b) When the ball hits the ground the 1st time, it has traveled a total vertical distance of 10ft.

When it hits the ground a 2nd time, a total of

$$10 + 2 \cdot \text{height of } 1^{st} \text{ bounce}$$

$$= 10 + 2 \cdot 10\left(\frac{3}{4}\right)$$

$$= 10 + 20\left(\frac{3}{4}\right)$$

At the 3rd time, a total of

$$10 + 20\left(\frac{3}{4}\right) + 2 \cdot \text{height of } 2^{nd} \text{ bounce}$$

$$= 10 + 20\left(\frac{3}{4}\right) + 2 \cdot 10\left(\frac{3}{4}\right)^2$$

$$= 10 + 20\left(\frac{3}{4}\right) + 20\left(\frac{3}{4}\right)^2$$

$$= 10\left(1 + 2\left(\frac{3}{4}\right) + 2\left(\frac{3}{4}\right)^2\right)$$

$$= 10\left(1 + 2\sum_{i=1}^{2}\left(\frac{3}{4}\right)^i\right) \quad (\text{using the } \Sigma \text{ notation})$$

(c) Following from (b) above, distance at

1^{st} hit is 10 ft

2^{nd} hit is $10 + 20\left(\frac{3}{4}\right)^1$ ft

3^{rd} hit is $10 + 20\left(\frac{3}{4}\right)^1 + 20\left(\frac{3}{4}\right)^2$ ft

4^{th} hit is $10 + 20\left(\frac{3}{4}\right)^1 + 20\left(\frac{3}{4}\right)^2 + 20\left(\frac{3}{4}\right)^3$ ft

$\vdots$

n^{th} hit is $10 + 20\left(\frac{3}{4}\right)^1 + 20\left(\frac{3}{4}\right)^2 + 20\left(\frac{3}{4}\right)^3 + \cdots + 20\left(\frac{3}{4}\right)^{n-1}$

$$= 10\left(1 + 2\left(\frac{3}{4}\right)^1 + 2\left(\frac{3}{4}\right)^2 + 2\left(\frac{3}{4}\right)^3 + \cdots + 2\left(\frac{3}{4}\right)^{n-1}\right)$$

$$= 10\left(1 + 2\sum_{i=1}^{n-1}\left(\frac{3}{4}\right)^i\right)$$ (using the Σ notation)

4. Write the first five terms of the following sequences.

(a) $a_1 = 1$, and $a_{n+1} = a_n + \frac{1}{2^n}$

(b) $a_1 = 2$, and $a_{n+1} = \frac{a_n}{2}(-1)^{n+1}$

(c) $a_1 = -2$, and $a_{n+1} = \frac{na_n}{n+1}$

(d) $a_1 = 2$, $a_2 = -1$, and $a_{n+2} = \frac{a_{n+1}}{a_n}$

(a) $a_1 = 1$, $a_{n+1} = a_n + \frac{1}{2^n}$

$$a_2 = a_1 + \frac{1}{2^1} = 1 + \frac{1}{2} = \frac{3}{2}$$

$$a_3 = a_2 + \frac{1}{2^2} = \frac{3}{2} + \frac{1}{4} = \frac{6+1}{4} = \frac{7}{4}$$

$$a_3 = a_2 + \frac{1}{2^2} = \frac{3}{2} + \frac{1}{4} = \frac{6+1}{4} = \frac{7}{4}$$

$$a_4 = a_3 + \frac{1}{2^3} = \frac{7}{4} + \frac{1}{8} = \frac{14+1}{8} = \frac{15}{8}$$

$$a_5 = a_4 + \frac{1}{2^4} = \frac{15}{8} + \frac{1}{16} = \frac{30+1}{16} = \frac{31}{16}$$

(b) $a_1 = 2$ and $a_{n+1} = \frac{a_n}{2}(-1)^{n+1}$

$$a_2 = \frac{a_1}{2}(-1)^2 = \frac{2}{2}(1) = 1$$

$$a_3 = \frac{a_2}{2}(-1)^3 = \frac{1}{2}(-1) = -\frac{1}{2}$$

$$a_4 = \frac{a_3}{2}(-1)^4 = \frac{-1/2}{2}(1) = -\frac{1}{4}$$

$$a_5 = \frac{a_4}{2}(-1)^5 = \frac{-1/4}{2}(-1) = \frac{1}{8}$$

(c) $a_1 = -2$, $a_{n+1} = \frac{n a_n}{n+1}$

$$a_2 = \frac{1 \cdot a_1}{2} = \frac{1(-2)}{2} = -1$$

$$a_3 = \frac{2a_2}{3} = \frac{2(-1)}{3} = -\frac{2}{3}$$

$$a_4 = \frac{3a_3}{4} = \frac{3(-2/3)}{4} = -\frac{1}{2}$$

$$a_5 = 4a_4 = 4(-1/2) = -\frac{2}{5}$$

$$a_5 = \frac{4a_4}{5} = \frac{4\left(-\frac{1}{2}\right)}{5} = -\frac{2}{5}$$

(d) $a_1 = 2,\quad a_2 = -1,\quad a_{n+2} = \dfrac{a_{n+1}}{a_n}$

$$a_3 = a_{1+2} = \frac{a_{1+1}}{a_1} = \frac{a_2}{a_1} = \frac{-1}{2}$$

$$a_4 = a_{2+2} = \frac{a_{2+1}}{a_2} = \frac{a_3}{a_2} = \frac{-\frac{1}{2}}{-1} = \frac{1}{2}$$

$$a_5 = a_{3+2} = \frac{a_{3+1}}{a_3} = \frac{a_4}{a_3} = \frac{\frac{1}{2}}{-\frac{1}{2}} = -1$$

5. Consider the sequence $\{a_n\}$ given by $a_n = \frac{4^n}{n!}$.

(a) Find $\lim_{n\to\infty} a_n$.

Here, we use the idea of the Squeeze Theorem by noticing that

$0 < \frac{4^n}{n!} = a_n$ for all n

and

$$a_1 = \frac{4^1}{1!} = 4$$

$$a_2 = \frac{4^2}{2!} = \frac{4\times4}{2\times1} = 8$$

$$a_3 = \frac{4^3}{3!} = \frac{4\times4\times4}{3\times2\times1} = \frac{32}{3}$$

$$4 \qquad 4\times4\times4\times4 = 4\cdot\frac{4\times4\times4}{3\times2\times1} = \frac{32}{3}$$

$$a_3 = \qquad 3! \qquad 3\times\ldots$$

$$a_4 = \frac{4^4}{4!} = \frac{4\times4\times4\times4}{4\times3\times2\times1} = \frac{4}{4}\cdot\frac{4\times4\times4}{3\times2\times1} = \frac{32}{3}$$

$$a_5 = \frac{4^5}{5!} = \frac{4\times4\times4\times4\times4}{5\times4\times3\times2\times1} = \frac{4}{5}\cdot\frac{4}{4}\cdot\frac{4\times4\times4}{3\times2\times1} = \frac{4}{5}\left(\frac{32}{3}\right)$$

$$a_6 = \frac{4^6}{6!} = \frac{4\times4\times4\times4\times4\times4}{6\times5\times4\times3\times2\times1} = \frac{4}{6}\cdot\frac{4}{5}\cdot\frac{4}{4}\cdot\frac{4\times4\times4}{3\times2\times1}$$

$$= \frac{4\cdot4}{6\cdot5}\left(\frac{32}{3}\right)$$

$$a_7 = \frac{4^7}{7!} = \frac{4\times4\times4\times4\times4\times4\times4}{7\times6\times5\times4\times3\times2\times1} = \frac{4}{7}\cdot\frac{4}{6}\cdot\frac{4}{5}\cdot\frac{4}{4}\cdot\frac{4\times4\times4}{3\times2\times1}$$

So in general,

$$a_n = \frac{4^n}{n!} = \frac{4}{n}\cdot\frac{4}{n-1}\cdot\frac{4}{n-2}\cdot\ldots\cdot\frac{4}{8}\cdot\frac{4}{7}\cdot\frac{4}{6}\cdot\frac{4}{5}\cdot\frac{4}{4}\cdot\frac{4\times4\times4}{3\times2\times1}$$

$$= \frac{4}{n}\cdot\left(\underbrace{\frac{4}{n-1}\cdot\frac{4}{n-2}\cdot\ldots\cdot\frac{4}{8}\cdot\frac{4}{7}\cdot\frac{4}{6}\cdot\frac{4}{5}}_{\text{each of the Terms is less than 1}}\right)\cdot\frac{4\times4\times4}{3\times2\times1}$$

$$< \frac{4}{n}\left(1\cdot1\cdots1\cdot1\cdot1\cdot1\cdot1\right)\cdot\frac{32}{3}$$

$$= \frac{128}{3n}$$

Thus,

$$0 < a_n < \frac{128}{\ldots}$$

Thus,

$$0 < a_n < \frac{128}{3n}$$

$$\Rightarrow \quad \lim_{n\to\infty} 0 \le \lim_{n\to\infty} a_n \le \lim_{n\to\infty} \frac{128}{3n}$$

$$\Rightarrow \quad 0 \le \lim_{n\to\infty} a_n \le \frac{128}{3} \lim_{n\to\infty} \frac{1}{n}$$

$$\Rightarrow \quad 0 \le \lim_{n\to\infty} a_n \le 0$$

$$\Rightarrow \quad \lim_{n\to\infty} a_n = 0$$

ie;

$$\lim_{n\to\infty} \frac{4^n}{n!} = 0.$$

6. Determine whether the sequences below converge or diverge. If a sequence converges, find the limit it converges to.

   (a) $a_n = \frac{\sin n}{n}$

   (b) $a_n = \int_1^n \frac{1}{x^p}\,dx$ for $p > 1$

   (c) $a_n = \frac{1}{n}\int_1^n \frac{1}{x}\,dx$

(a) $a_n = \frac{\sin n}{n}$

Since $|\sin x| \le 1$, it follows that

$$|\sin n| \le 1 \iff -1 \le \sin n \le 1$$

$$\Leftrightarrow \quad -\frac{1}{n} \le \frac{\sin n}{n} \le \frac{1}{n}$$

So

$$\lim_{n\to\infty}\left(\frac{-1}{n}\right) \le \lim_{n\to\infty} \frac{\sin n}{n} \le \lim_{n\to\infty} \frac{1}{n}$$

$$\Rightarrow \quad -\lim_{n\to\infty} \frac{1}{n} \le \lim_{n\to\infty} \frac{\sin n}{n} \le \lim_{n\to\infty} \frac{1}{n}$$

$$\Rightarrow \quad -(0) \le \lim_{n\to\infty} \frac{\sin(n)}{n} \le 0$$

$$\Rightarrow \quad 0 \le \lim_{n\to\infty} \frac{\sin n}{n} \le 0$$

$$\Rightarrow \quad \lim_{n\to\infty} \frac{\sin n}{n} = 0.$$

6 $\quad a_n = \int_1^n \frac{1}{x^p}\,dx, \quad p>1$

$$a_n = \int_1^n x^{-p}\,dx$$

$$= \left.\frac{x^{-p+1}}{-p+1}\right|_1^n$$

$$= \frac{1}{1-p}\left(x^{1-p}\right)\Big|_1^n$$

$$= \frac{1}{1-p}\left(n^{1-p}-1\right)$$

So

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{1-p}\left(n^{1-p}-1\right)$$

$$= \frac{1}{1-p}\left(\lim_{n\to\infty} n^{1-p} - 1\right)$$

$$= \frac{1}{1-p}\left(0-1\right)$$ since $p>1 \Rightarrow 1-p<0$.

$$= \frac{-1}{1-p} = \frac{1}{p-1}$$

ⓒ $a_n = \frac{1}{n}\int_1^n \frac{1}{x}\,dx$

$$= \frac{1}{n}\ln x\Big|_1^n$$

$$= \frac{1}{n}\left(\ln n - \ln 1\right)$$

$$= \frac{\ln n}{n}$$

So

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\ln n}{n}$$

$$= 0$$ since $\lim_{x\to\infty} \frac{\ln x}{x} \stackrel{L'H}{=} \lim_{x\to\infty} \frac{1/x}{x}$

$$= \lim_{x\to\infty} \frac{1}{x^2}$$

$= 0.$