Series

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1. Consider the sequence given by $a_1 = 1$, $a_n = \left(1 - \frac{1}{n^2}\right) a_{n-1}$.

   (a) Write out the first 4 terms of the sequence.

   (b) Determine whether the sequence $\{a_n\}$ converges. (Challenge: If it does, find $\lim_{n\to\infty} a_n$.)

(a) $a_1 = 1$, $a_n = \left(1 - \frac{1}{n^2}\right) a_{n-1}$

So,

$$a_2 = \left(1 - \frac{1}{2^2}\right) a_{2-1} = \left(1 - \frac{1}{4}\right) a_1 = \left(1 - \frac{1}{4}\right) 1 = \frac{3}{4},$$

$$a_3 = \left(1 - \frac{1}{3^2}\right) a_{3-1} = \left(1 - \frac{1}{9}\right) a_2 = \left(\frac{9-1}{9}\right) \frac{3}{4} = \frac{8}{9} \cdot \frac{3}{4} = \frac{2}{3},$$

$$a_4 = \left(1 - \frac{1}{4^2}\right) a_{4-1} = \left(1 - \frac{1}{16}\right) a_3 = \left(\frac{16-1}{16}\right) \frac{2}{3} = \frac{15}{16} \cdot \frac{2}{3} = \frac{5}{8}.$$

(b)
$$a_n = \left(1 - \frac{1}{n^2}\right) a_{n-1}$$
$$= \left(1 - \frac{1}{n^2}\right)\left(\left(1 - \frac{1}{(n-1)^2}\right) a_{n-2}\right)$$
$$= \left(1 - \frac{1}{n^2}\right)\left(1 - \frac{1}{(n-1)^2}\right)\left(\left(1 - \frac{1}{(n-2)^2}\right) a_{n-3}\right)$$
$$\vdots$$
$$= \left(1 - \frac{1}{n^2}\right)\left(1 - \frac{1}{(n-1)^2}\right)\left(1 - \frac{1}{(n-2)^2}\right) \cdots \left(1 - \frac{1}{(n-(n-2))^2}\right) a_{n-(n-1)}$$
$$= \left(1 - \frac{1}{n^2}\right)\left(1 - \frac{1}{(n-1)^2}\right)\left(1 - \frac{1}{(n-2)^2}\right) \cdots \left(1 - \frac{1}{2^2}\right)$$

$$= \left(1-\frac{1}{n^2}\right)\left(1-\frac{1}{(n-1)^2}\right)\left(1-\frac{1}{(n-2)^2}\right)\cdots\left(1-\frac{1}{2^2}\right)$$ since $a_1 = 1$

$$= \left(\frac{n^2-1}{n^2}\right)\left(\frac{(n-1)^2-1}{(n-1)^2}\right)\left(\frac{(n-2)^2-1}{(n-2)^2}\right)\cdots\left(\frac{2^2-1}{2^2}\right)$$

$$= \left(\frac{(n+1)(n-1)}{n^2}\right)\left(\frac{n(n-2)}{(n-1)^2}\right)\left(\frac{(n-1)(n-3)}{(n-2)^2}\right)\cdots\left(\frac{3(1)}{2^2}\right)$$

$$= \frac{\big((n+1)n(n-1)\cdots 3\big)\big((n-1)(n-2)(n-3)\cdots 1\big)}{n^2(n-1)^2(n-2)^2\cdots 2^2}$$

$$= \frac{1}{2}\left(\frac{\big((n+1)n(n-1)\cdots 3\times 2\big)\big((n-1)(n-2)(n-3)\cdots 1\big)}{\big(n(n-1)(n-2)\cdots 2\big)^2}\right)$$

$$= \frac{1}{2}\left(\frac{(n+1)!\,(n-1)!}{(n!)^2}\right)$$

$$= \frac{1}{2}\left(\frac{(n+1)\,n!\,(n-1)!}{n!\,n!}\right)$$

$$= \frac{1}{2}\left(\frac{(n+1)\,(n-1)!}{n(n-1)!}\right)$$

$$= \frac{1}{2}\left(\frac{n+1}{n}\right)$$

So

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{2}\left(\frac{n+1}{n}\right)$$

$$= \frac{1}{2} \lim_{n\to\infty} \left(\frac{n+1}{n}\right)$$

$$= \frac{1}{2} \lim_{n\to\infty} \left(\frac{1+\frac{1}{n}}{1}\right)$$

$$= \frac{1}{2}(1)$$

$$= \frac{1}{2}$$

Hence, the sequence converges and $\lim_{n\to\infty} a_n = \frac{1}{2}$.

2. A ball is dropped from a height of ten feet and bounces. Each bounce is $\frac{3}{4}$ the height of the bounce before. So, after the ball hits the floor for the first time, the ball rises to a height of $10(\frac{3}{4}) = 7.5$ feet, and after it hits the floor for a second time, it rises to a height of $7.5(\frac{3}{4}) = 10(\frac{3}{4})^2 = 5.625$ feet.

   (a) What height does the ball rise to after it hits the floor for the nth time?

   (b) Find an expression for the total vertical distance the ball has travelled when it hits the ground for the first, second, and third time.

   (c) Find an expression for the total vertical distance the ball has travelled when it hits the ground for the nth time.

(a) $a_1 = 10\left(\frac{3}{4}\right)$ (1^{st} bounce)

$a_2 = \frac{3}{4}a_1 = \frac{3}{4}\cdot 10\left(\frac{3}{4}\right) = 10\left(\frac{3}{4}\right)^2$ (2^{nd} bounce)

$a_3 = \frac{3}{4}a_2 = \frac{3}{4}\cdot 10\left(\frac{3}{4}\right)^2 = 10\left(\frac{3}{4}\right)^3$ (3^{rd} bounce)

$a_4 = \frac{3}{4}a_3 = \frac{3}{4}\cdot 10\left(\frac{3}{4}\right)^3 = 10\left(\frac{3}{4}\right)^4$ (4^{th} bounce)

$$a_4 = \frac{3}{4} a_3 = \frac{3}{4} \cdot 10\left(\frac{3}{4}\right)^3 = 10\left(\frac{3}{4}\right)^4 \quad (4^{th} \text{ bounce})$$

$$\vdots$$

$$a_n = \frac{3}{4}\left(\frac{3}{4}\right)^n \quad (n^{th} \text{ bounce})$$

(b) When the ball hits the ground the 1st time, it has traveled a total vertical distance of 10ft.

When it hits the ground a 2nd time, a total of

$$10 + 2 \cdot \text{height of } 1^{st} \text{ bounce}$$

$$= 10 + 2 \cdot 10\left(\frac{3}{4}\right)$$

$$= 10 + 20\left(\frac{3}{4}\right)$$

At the 3rd time, a total of

$$10 + 20\left(\frac{3}{4}\right) + 2 \cdot \text{height of } 2^{nd} \text{ bounce}$$

$$= 10 + 20\left(\frac{3}{4}\right) + 2 \cdot 10\left(\frac{3}{4}\right)^2$$

$$= 10 + 20\left(\frac{3}{4}\right) + 20\left(\frac{3}{4}\right)^2$$

$$= 10\left(1 + 2\left(\frac{3}{4}\right) + 2\left(\frac{3}{4}\right)^2\right)$$

$$= 10\left(1 + 2\sum_{i=1}^{2}\left(\frac{3}{4}\right)^i\right) \quad (\text{using the } \Sigma \text{ notation})$$

ⓒ Following from ⓑ above, distance at

1^{st} hit is 10 ft

2^{nd} hit is $10 + 20\left(\frac{3}{4}\right)^1$ ft

3^{rd} hit is $10 + 20\left(\frac{3}{4}\right)^1 + 20\left(\frac{3}{4}\right)^2$ ft

4^{th} hit is $10 + 20\left(\frac{3}{4}\right)^1 + 20\left(\frac{3}{4}\right)^2 + 20\left(\frac{3}{4}\right)^3$ ft

$\vdots$

n^{th} hit is $10 + 20\left(\frac{3}{4}\right)^1 + 20\left(\frac{3}{4}\right)^2 + 20\left(\frac{3}{4}\right)^3 + \cdots + 20\left(\frac{3}{4}\right)^{n-1}$

$$= 10\left(1 + 2\left(\frac{3}{4}\right)^1 + 2\left(\frac{3}{4}\right)^2 + 2\left(\frac{3}{4}\right)^3 + \cdots + 2\left(\frac{3}{4}\right)^{n-1}\right)$$

$$= 10\left(1 + 2\sum_{i=1}^{n-1}\left(\frac{3}{4}\right)^i\right)$$ (using the Σ notation)

$$= 10\left(1 + 2\left(\frac{\frac{3}{4}\left(1-\left(\frac{3}{4}\right)^n\right)}{1-\frac{3}{4}}\right)\right)$$ using the partial sum of geometric series

$$= 10\left(1 + 2\left(\frac{\frac{3}{4}\left(1-\left(\frac{3}{4}\right)^n\right)}{1/4}\right)\right)$$

$$= 10\left(1 + 6\left(1-\left(\frac{3}{4}\right)^n\right)\right)$$

$$= 10\left(1 + 6 - 6\left(\frac{3}{4}\right)^n\right)$$

$$= 70 - 60\left(\frac{3}{4}\right)^n$$

$$= 70 - 60\left(\frac{3}{4}\right)^n$$

3. Consider the sequence $\{a_n\}$ given by $a_n = \frac{4^n}{n!}$.

   (a) Show that a_n is eventually decreasing. Conclude that $\lim_{n\to\infty} a_n$ exists.

   (b) Find this limit.

(b) In the last class, we showed that

$$\lim_{n\to\infty} \frac{4^n}{n!} = 0.$$

(a) We can know before hand that $\{a_n\}$ converges even without finding the limit (using the Monotonic convergence theorem) since

$$0 < \frac{4^n}{n!} < \frac{128}{3n} \quad \text{for all } n \geq 1$$

$$\Rightarrow 0 < \frac{4^n}{n!} < \frac{128}{3} \quad \text{for all } n \geq 1$$

$$\Rightarrow \frac{4^n}{n!} \text{ is bounded}$$

and

$$a_{n+1} = \frac{4^{n+1}}{(n+1)!} = \frac{4^n \cdot 4}{(n+1)n!} = \frac{4}{(n+1)} \cdot \frac{4^n}{n!} = \frac{4}{(n+1)} a_n$$

$$\Rightarrow \quad a_{n+1} = \frac{4}{(n+1)} a_n < a_n \quad \text{for all } n \geq 4.$$

$$\Rightarrow \quad a_{n+1} < a_n$$

$\therefore \{a_n\}$ is monotonic.

$$\Rightarrow \{a_n\} \text{ is monotonic.}$$

So by the Monotonic Convergence Theorem $\{a_n\}$ converges since it is bounded and monotonic.

4. Rewrite the sum using Sigma (Σ) notation.

   (a) $x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots$

   (b) $1 + 2x + 2^2\frac{x^2}{2!} + 2^3\frac{x^3}{3!} + 2^4\frac{x^4}{4!} + \cdots$

   (c) $\frac{1}{1-x}$ provided that $|x| < 1$

(a) $$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots = \sum_{i=1}^{\infty} (-1)^{i-1} \frac{x^{2i-1}}{(2i-1)!} = \sum_{j=0}^{\infty} (-1)^j \frac{x^{2j+1}}{(2j+1)!}$$

where $j = i-1$

(b) $$1 + 2x + 2^2\frac{x^2}{2!} + 2^3\frac{x^3}{3!} + 2^4\frac{x^4}{4!} + \cdots = \sum_{i=0}^{\infty} \frac{(2x)^i}{i!}$$

This is the sum to infinity of a geometric series with $a=1$ and $r=x$.

(c) $\frac{1}{1-x}$ provided that $|x|<1$

$$= 1 + x + x^2 + x^3 + x^4 + \cdots$$

$$= \sum_{i=0}^{\infty} x^i$$

5. On New Year's Day, you begin by saving one penny on the first day, two pennies on the second day, three pennies on the third day, and so forth. How many days until you've saved \$200?

On the first day you saved 1 penny

On the second day you saved 2 pennies

On the third day you saved 3 pennies

$\vdots \qquad\qquad\qquad\qquad\qquad \vdots$

On the nth day you saved n pennies

$\Rightarrow$ After n days, you saved

$$1+2+3+4+\cdots+n = \sum_{i=1}^{n} i = \frac{n(n+1)}{2} \text{ pennies.}$$

So at what n (day) is

$$\frac{n(n+1)}{2} = \$200 = 20000 \text{ pennies?}$$

$$\Rightarrow \quad n^2+n-40000 = 0$$

$$\Rightarrow \quad n = \frac{-1 \pm \sqrt{1-4(1)(-40000)}}{2(1)}$$

$$= \frac{-1 \pm \sqrt{160001}}{2}$$

$$= \frac{-1+400.00125}{2}, \quad \frac{-1-400.00125}{2}$$

$$= 199.500625, \quad -200.500625$$

Since # of days can't negative and $n = 199.5$ means 199 days + half a day, it follows that you need 200 days to have saved at least \$200.

6. Consider the sequence $\{a_n\}$ with partial sums $\sum_{n=1}^{N} a_n = S_N$. Suppose the sequence $\{S_N\}$ satisfies $S_N = \dfrac{\sqrt{N+2}-1}{(N+1)^2}$.

(a) Find $\sum_{n=1}^{\infty} a_n$.

(b) Evaluate $\lim_{n\to\infty} a_n$.

(c) Find a closed formula for a_n (leave your answer unsimplified).

ⓐ $\sum_{n=1}^{\infty} a_n = \lim_{N\to\infty} S_N = \lim_{N\to\infty} \dfrac{\sqrt{N+2}-1}{(N+1)^2}$

$$= \lim_{N\to\infty} \frac{\sqrt{N+2}-1}{N^2\left(1+\frac{2}{N}\right)^2}$$

$$= \lim_{N\to\infty} \frac{\frac{1}{N^2}\left(\sqrt{N+2}-1\right)}{\left(1+\frac{2}{N}\right)^2}.$$

$$= \lim_{N\to\infty} \frac{\sqrt{\frac{N}{N^4}+\frac{2}{N^4}}-\frac{1}{N^2}}{\left(1+\frac{2}{N}\right)^2}$$

$$= \lim_{N\to\infty} \frac{\sqrt{\frac{1}{N^3}+\frac{2}{N^4}}-\frac{1}{N^2}}{\left(1+\frac{2}{N}\right)^2}$$

$$= \frac{\sqrt{0+0}-0}{}$$

$$= \frac{\sqrt{0+0} - 0}{(1+0)^2}$$

$$= 0$$

(b) $\sum_{n=1}^{\infty} a_n$ converges $\Rightarrow \lim_{n\to\infty} a_n = 0$

(c) Since

$$S_n = a_1 + a_2 + a_3 + \cdots + a_{n-1} + a_n$$

$$= (a_1 + a_2 + a_3 + \cdots + a_{n-1}) + a_n$$

$$= S_{n-1} + a_n,$$

$$a_n = S_n - S_{n-1}$$

$$= \frac{\sqrt{n+2} - 1}{(n+1)^2} - \frac{\sqrt{(n-1)+2} - 1}{(n-1+1)^2}$$

$$= \frac{\sqrt{n+2} - 1}{(n+1)^2} - \frac{\sqrt{n+1} - 1}{n^2}$$

$$=$$

7. Evaluate the following sums.

(a) $\sum_{k=1}^{\infty} \tan^{-1}(k+1) - \tan^{-1}(k)$

(b) $\sum_{i=1}^{\infty} \ln(i+1) - \ln(i)$

(c) $\sum_{j=1}^{\infty} \frac{a}{\sqrt{j+1}} - \frac{a}{\sqrt{j}}$

(d) $\sum_{i=1}^{\infty} \frac{3}{i^2+2i}$

(a) $$\sum_{k=1}^{\infty}\left(\tan^{-1}(k+1) - \tan^{-1}(k)\right) = \lim_{n\to\infty} \sum_{k=1}^{n}\left(\tan^{-1}(k+1) - \tan^{-1}(k)\right)$$

$$= \lim_{n\to\infty}\Big(\left(\cancel{\tan^{-1}2} - \tan^{-1}1\right) + \left(\cancel{\tan^{-1}3} - \cancel{\tan^{-1}2}\right) + \left(\cancel{\tan^{-1}4} - \cancel{\tan^{-1}3}\right) + \cdots + \left(\tan^{-1}(n+1) - \cancel{\tan^{-1}n}\right)\Big)$$

$$= \lim_{n\to\infty}\left(-\tan^{-1}1 + \tan^{-1}(n+1)\right)$$

$$= \lim_{n\to\infty}\left(\tan^{-1}(n+1) - \tan^{-1}1\right)$$

$$= \lim_{n\to\infty} \tan^{-1}(n+1) - \frac{\pi}{4}$$

$$= \frac{\pi}{2} - \frac{\pi}{4}$$

$$= \underline{\underline{\frac{\pi}{4}}}$$

ⓑ $\sum_{i=1}^{\infty}\left(\ln(i+1)-\ln(i)\right) = \lim_{n\to\infty} \sum_{i=1}^{n} \left(\ln(i+1)-\ln(i)\right)$

$$= \lim_{n\to\infty} \left((\ln 2 - \ln 1) + (\ln 3 - \ln 2) + (\ln 4 - \ln 3) + \cdots + (\ln(n+1) - \ln n) \right)$$

$$= \lim_{n\to\infty} \left((\cancel{\ln 2} - \ln 1) + (\cancel{\ln 3} - \cancel{\ln 2}) + (\cancel{\ln 4} - \cancel{\ln 3}) + \cdots + (\ln(n+1) - \cancel{\ln n}) \right)$$

$$= \lim_{n\to\infty} \left(-\ln 1 + \ln(n+1) \right)$$

$$= \lim_{n\to\infty} \ln(n+1)$$

$$= \infty$$

The sum diverges.

ⓒ $\sum_{i=1}^{\infty}\left(\frac{a}{\sqrt{j+1}} - \frac{a}{\sqrt{j}}\right) = \lim_{n\to\infty} \sum_{j=1}^{n} \left(\frac{a}{\sqrt{j+1}} - \frac{a}{\sqrt{j}}\right)$

$$= \lim_{n\to\infty} \left(\left(\frac{a}{\sqrt{2}} - \frac{a}{\sqrt{1}}\right) + \left(\frac{a}{\sqrt{3}} - \frac{a}{\sqrt{2}}\right) + \left(\frac{a}{\sqrt{4}} - \frac{a}{\sqrt{3}}\right) + \cdots + \left(\frac{a}{\sqrt{n+1}} - \frac{a}{\sqrt{n}}\right) \right)$$

$$= \lim_{n\to\infty} \left(\left(\cancel{\frac{a}{\sqrt{2}}} - \frac{a}{\sqrt{1}}\right) + \left(\cancel{\frac{a}{\sqrt{3}}} - \cancel{\frac{a}{\sqrt{2}}}\right) + \left(\cancel{\frac{a}{\sqrt{4}}} - \cancel{\frac{a}{\sqrt{3}}}\right) + \cdots + \left(\frac{a}{\sqrt{n+1}} - \cancel{\frac{a}{\sqrt{n}}}\right) \right)$$

$$= \lim_{n\to\infty} \left(-\frac{a}{\sqrt{1}} + \frac{a}{\sqrt{n+1}} \right)$$

$$= \lim_{n\to\infty} \left(-\frac{a}{\sqrt{1}} + \frac{a}{\sqrt{n+1}} \right)$$

$$= -a + a \lim_{n\to\infty} \frac{1}{\sqrt{n+1}}$$

$$= -a + a \lim_{n\to\infty} \frac{1/n}{\sqrt{1+\frac{1}{n}}}$$

$$= -a + a\left(\frac{0}{1}\right)$$

$$= \underline{\underline{-a}}$$

d) $$\sum_{i=1}^{\infty} \frac{3}{i^2+2i} = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{3}{i(i+2)}$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{3/2}{i} - \frac{3/2}{i+2} \right)$$ by partial fractions

$$= \lim_{n\to\infty} \frac{3}{2} \sum_{i=1}^{n} \left(\frac{1}{i} - \frac{1}{i+2} \right)$$

$$= \frac{3}{2} \lim_{n\to\infty} \Bigg(\left(\frac{1}{1} - \cancel{\frac{1}{3}}\right) + \left(\frac{1}{2} - \cancel{\frac{1}{4}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{5}}\right) + \left(\cancel{\frac{1}{4}} - \cancel{\frac{1}{6}}\right) + \left(\cancel{\frac{1}{5}} - \frac{1}{7}\right)$$

$$+ \left(\cancel{\frac{1}{6}} - \cancel{\frac{1}{8}}\right) + \cdots + \left(\cancel{\frac{1}{n-5}} - \cancel{\frac{1}{n-3}}\right) + \left(\cancel{\frac{1}{n-4}} - \frac{1}{n-2}\right) + \left(\cancel{\frac{1}{n-3}} - \cancel{\frac{1}{n-1}}\right)$$

$$+ \left(\frac{1}{n-2} - \cancel{\frac{1}{n}}\right) + \left(\cancel{\frac{1}{n-1}} - \frac{1}{n+1}\right) + \left(\cancel{\frac{1}{n}} - \frac{1}{n+2}\right) \Bigg)$$

$$= \frac{3}{2} \lim_{n\to\infty} \left(\frac{1}{1} + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2} \right)$$

$$= \frac{3}{2} \lim_{n\to\infty} \left(\frac{3}{2} - \frac{1}{n+1} - \frac{1}{n+2} \right)$$

$$= \frac{3}{2} \left(\frac{3}{2} - \lim_{n\to\infty} \frac{1}{n+1} - \lim_{n\to\infty} \frac{1}{n+2} \right)$$

$$= \frac{3}{2} \left(\frac{3}{2} - 0 - 0 \right)$$

$$= \frac{9}{4}$$

8. Calculate the infinite sum $0.4 + 0.16 + 0.064 + 0.0256 + \cdots$

$0.4 + 0.16 + 0.064 + 0.0256 + \cdots$

$$= \frac{4}{10} + \frac{16}{10^2} + \frac{64}{10^3} + \frac{256}{10^4} + \cdots$$

$$= \frac{4}{10} \left(1 + \frac{4}{10} + \frac{16}{10^2} + \frac{64}{10^3} + \cdots \right)$$

$$= \frac{4}{10} \left(1 + \frac{4}{10} + \frac{4^2}{10^2} + \frac{4^3}{10^3} + \cdots \right)$$

$$= \frac{4}{10} \left(1 + \sum_{i=1}^{\infty} \left(\frac{4}{10} \right)^i \right)$$

geometric series with $a = \frac{4}{10}, r = \frac{4}{10}$

$$= \frac{4}{10}\left(1 + \frac{4/10}{1 - 4/10}\right)$$

$$= \frac{4}{10}\left(1 + \frac{4/10}{6/10}\right)$$

$$= \frac{4}{10}\left(1 + \frac{4}{6}\right)$$

$$= \frac{4}{10}\left(\frac{10}{6}\right) = \underline{\underline{\frac{2}{3}}}$$

9. Find a formula for the nth partial sum of each series. *Hint for (b):* Write as a telescoping series.

(a) $2 + \frac{2}{3} + \frac{2}{9} + \frac{2}{27} + \cdots + \frac{2}{3^n} + \cdots$

(b) $\frac{5}{1\cdot 2} + \frac{5}{2\cdot 3} + \frac{5}{3\cdot 4} + \cdots + \frac{5}{n\cdot(n+1)} + \cdots$

(a) $2 + \frac{2}{3} + \frac{2}{9} + \frac{2}{27} + \cdots + \frac{2}{3^n} + \cdots$

$$= 2\left(1 + \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \cdots + \frac{1}{3^n} + \cdots\right)$$

$$= 2\left(1 + \sum_{i=1}^{\infty}\left(\frac{1}{3}\right)^i\right)$$

So the nth partial sum

$$S_n = 2\left(1 + \sum_{i=1}^{n}\left(\frac{1}{3}\right)^i\right)$$

$$= 2\left(1 + \frac{\frac{1}{3}\left(1 - \left(\frac{1}{3}\right)^n\right)}{1 - \frac{1}{3}}\right)$$

$$= 2\left(1 + \frac{\frac{1}{3}\left(1 - \left(\frac{1}{3}\right)^n\right)}{2/3}\right)$$

$$= 2\left(1 + \frac{1}{2}\left(1 - \left(\frac{1}{3}\right)^n\right)\right)$$

$$= 2\left(1 + \frac{1}{2} - \frac{1}{2}\left(\frac{1}{3}\right)^n\right)$$

$$= 2\left(\frac{3}{2} - \frac{1}{2}\left(\frac{1}{3}\right)^n\right)$$

$$= 3 - \left(\frac{1}{3}\right)^n$$

(b) $$\frac{5}{1\cdot 2} + \frac{5}{2\cdot 3} + \frac{5}{3\cdot 4} + \cdots + \frac{5}{n(n+1)} + \cdots$$

$$= \sum_{i=1}^{\infty} \frac{5}{i(i+1)}$$

$$= \sum_{i=1}^{\infty} \left(\frac{5}{i} - \frac{5}{i+1}\right)$$

$$= 5 \sum_{i=1}^{\infty} \left(\frac{1}{i} - \frac{1}{i+1}\right)$$

So the nth partial sum

$$S_n = 5 \sum_{i=1}^{n} \left(\frac{1}{i} - \frac{1}{i+1}\right)$$

$$= 5\Bigg(\left(\frac{1}{1} - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}\right) + \left(\cancel{\frac{1}{4}} - \cancel{\frac{1}{5}}\right) + \cdots + \left(\cancel{\frac{1}{n-4}} - \cancel{\frac{1}{n-3}}\right) + \left(\cancel{\frac{1}{n-3}} - \frac{1}{n-2}\right)$$
$$+ \left(\frac{1}{n-2} - \cancel{\frac{1}{n-1}}\right) + \left(\cancel{\frac{1}{n-1}} - \cancel{\frac{1}{n}}\right) + \left(\cancel{\frac{1}{n}} - \frac{1}{n+1}\right)\Bigg)$$

$$= 5\left(\frac{1}{1} - \frac{1}{n+1}\right)$$

$$= 5\left(\frac{n+1-1}{n+1}\right)$$

$$= \frac{5n}{n+1}$$

For Geometric series:

* If the sequence is given as $a_1, a_2, a_3, \cdots, a_n$, then

$$\left.\begin{aligned} S_n &= a_1 + a_2 + a_3 + a_4 + a_5 + \cdots + a_n \\ &= a + ar + ar^2 + ar^3 + ar^4 + \cdots + ar^{n-1} \end{aligned}\right\} \text{both have } n \text{ terms} \quad \text{—— ①}$$

$$① \times r \Rightarrow rS_n = ar + ar^2 + ar^3 + ar^4 + \cdots + ar^{n-1} + ar^n \quad \text{—— ②}$$

$$① - ② \Rightarrow S_n - rS_n = a - ar^n$$

$$\Rightarrow (1-r)S_n = a(1-r^n)$$

$$\Rightarrow \boxed{S_n = \frac{a(1-r^n)}{1-r}} \quad \text{—— } (*)$$

* If the sequence is given as $a_0, a_1, a_2, \cdots, a_n$, then

$$\left.\begin{aligned} S_n &= a_0 + a_1 + a_2 + a_3 + a_4 + \cdots + a_n \\ &= a + ar + ar^2 + ar^3 + ar^4 + \cdots + ar^n \end{aligned}\right\} \text{both have } n+1 \text{ terms} \quad \text{—— ③}$$

$③ \times r \Rightarrow rS_n = \quad ar + ar^2 + ar^3 + ar^4 + \cdots + ar^n + ar^{n+1}$ —— ④

$③ - ④ \Rightarrow S_n - rS_n = a - ar^{n+1}$

$\Rightarrow (1-r)S_n = a(1-r^{n+1})$

$\Rightarrow \boxed{S_n = \frac{a(1-r^{n+1})}{1-r}}$ ———— (**)

I admit I fell victim of mixing (*) and (**) in class today!