Questions for recitation 14 April 2021

1. Find the Taylor series for the following functions about the indicated center a:

(a)	$f(x) = e^{-2x}$	$a = 0$
(b)	$f(x) = x^2 - 2x + 4$	$a = b$
(c)	$f(x) = \frac{1}{x^2}$	$a = 1$
(d)	$f(x) = 2^x$	$a = 1$

Solution:

(a) By direct substitution into $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$: $f(x) = e^{-2x} = \sum_{n=0}^{\infty}(-1)^n \frac{2^n x^n}{n!}$

(b) This is a polynomial already! Its Taylor series is an attempt to approximate it with a polynomial; here, we're approximating $f(x)$ with $x^2 - 2x + 4$, which is effective. (We could recenter in terms of $(x - b)$, but who cares?)

(c) By taking derivatives: $f(1) = 1, f'(x) = \frac{-2}{x^3}, f^{(n)}(x) = \frac{(-1)^n(n+1)!}{x^{n+2}}$. Then the series is:

$$f(x) = \sum_{n=0}^{\infty}(-1)^n(n+1)(x-1)^n.$$

The root test will demonstrate an IOC of $(0, 2)$.

(d) The derivatives are $f^{(n)}(x) = (\ln 2)^n 2^x$, so $f(x) = \sum_{n=0}^{\infty} \frac{(\ln 2)^n 2}{n!}(x-1)^n$ (We could have found the Maclaurin series more easily by a substitution using $2^x = e^{x \ln 2}$).

2. Consider the following steps for estimating $\int_0^1 \frac{dx}{1+x^4}$.

(a) Calculate the first 4 non-zero terms of the Maclaurin series for the integrand.

(b) Determine the radius of convergence of this series.

(c) Using the series, evaluate the integral.

(d) How should we bound the error involved in this integral?

Solution:

(a) By substitution into the geometric with $r = -x^4$:

$$\frac{1}{1+x^4} = 1 - x^4 + x^8 - x^{12} + \ldots.$$

(b) $|-x^4| < 1 \implies |x| < 1$ is our interval of convergence, so $R = 1$.

(c) $I \approx x - \frac{x^5}{5} + \frac{x^9}{9} - \frac{x^{13}}{13}]_0^1 = 1 - \frac{1}{5} + \frac{1}{9} - \frac{1}{13}$.

(d) Rather than figuring out how to pass the error bound from Taylor Remainder Theorem in (a) through the integrand, we can just use the Alternating Series Estimation Theorem, and bound it by the "next" term in the sequence of $\frac{1}{17}$.

3. Find the Maclaurin series for $f(x) = \dfrac{x^2}{1+x^2}$. For what values does it converge? **Solution:** By substitution into the geometric with $r = -x^2$:

$$f(x) = x^2 \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n+2}.$$

The interval relies on the geometric, so $x^2 < 1 \implies -1 < x < 1$. We can easily observe that $x = 1, x = -1$ won't work, as in either case the series will be $1 - 1 + 1 - 1 + \dots$.

4. Find the sum of the following series by starting with a similar Taylor series and performing any necessary transformations.

   (a) $x + x^3 + \dfrac{x^5}{2!} + \dfrac{x^7}{3!} + \dots$

   (b) $2 - 3 \cdot 2x + 4 \cdot 3x^2 - 5 \cdot 4x^3 + \dots$

   (c) $\dfrac{x^2}{2} + \dfrac{x^4}{4} + \dfrac{x^6}{6} + \dfrac{x^8}{8} + \dots$

   (d) $1 - \dfrac{3x^2}{2!} + \dfrac{5x^4}{4!} - \dfrac{7x^6}{6!} + \dots$.

   Solution: This is a tricky approach to Taylor series problems, and there are a lot of ways to get started on them. Try to write down the formula for a_n and compare it to known series: the geometric, exponential, and trigonometric series.

   (a) Noting the factorial denominators and that each term is positive, this looks like a similar form to the series for $e^x = \sum_{n=0}^{\infty} x^n$. Noting that $e^{2x} = 1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!}$, we observe that this series is the series for xe^{x^2}.

   (b) The coefficent of each term looks like $(-1)^n(n+1)(n+2)$ if we start from $n = 0$. This looks like we took a couple derivatives! In fact, we can get this series by twice differentiation of $x^2 - x^3 + x^4 - x^5 + \dots$, but this is the series for $\dfrac{x^2}{1+x}$. So we can differentiate this function twice, and arrive at the function corresponding to the given series, or:

$$\sum_{n=0}^{\infty} (-1)^n (n+2)(n+1)x^n = \sum_{n=0}^{\infty} \frac{d^2}{dx^2}(-x)^{n+2} = \frac{d^2}{dx^2}\left(\frac{x^2}{1+x}\right) = \frac{2}{(1+x)^3}$$

   (c) The nth term of the sequence is $\frac{x^{2n}}{2n}$, which is the antiderivative of x^{2n-1}. Let's use this to motivate our answer:

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{x^{2n}}{2n} &= \sum_{n=1}^{\infty} \int x^{2n-1}\,dx \\
&= \int x \sum_{n=0}^{\infty} x^{2n}\,dx \\
&= \int x \frac{1}{1-x^2}\,dx \qquad \text{(geo series)} \\
&= \int \frac{x}{1-x^2}\,dx = \frac{-1}{2}\ln(1-x^2)
\end{aligned}$$

We also could have arrived here by modifying the series for $\ln(1-x) = -\sum \frac{x^n}{n}$.

(d) The nth term is given by $a_n = \dfrac{(-1)^n(2n+1)x^{2n}}{(2n)!}$. This looks pretty messy, but we can consider looking at series with similar forms. The alternating signs and exponential denominators suggest that we should consider series for sin and cos. Let's try to work backwards and get our function into the same form as the series for a trig function. First, note that $\displaystyle\int a_n\,dx = \frac{(-1)^n x^{2n+1}}{2n} = x\frac{(-1)^n x^{2n}}{2n}$ where now the exponents match the factorials, as in the series for sin and cos. In fact, we have arrived at:

$$\sum_{n=0}^{\infty} \int a_n\,dx = x \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n} = x\cos(x).$$

So if we differentiate this equation, we conclude that the original series was the derivative of the series for $x\cos(x)$, so it was $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n(2n+1)x^{2n}}{(2n)!} = \cos(x) - x\sin(x)$.

5. Determine whether the following series converge or diverge:

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^2+1}$

(b) $\displaystyle\sum_{n=1}^{\infty} \ln(2n) - \sum_{n=1}^{\infty} \ln(4n+2)$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n}$

Solution:

(a) Diverges by limit comparison test to $b_n = \frac{1}{n}$.

(b) Diverges because each sum diverges by test for divergence. Even if we *could* combine the $\infty - \infty$ into a single sum (you can't do that!!), the result would still diverge by the test for divergence, as $\displaystyle\lim_{n\to\infty} \ln(2n) - \ln(4n+2) \to \ln(.5) \neq 0$.

(c) Ratio test:

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)!}{n!} \cdot \frac{n^n}{(n+1)^{n+1}} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n = \frac{1}{e} < 1,$$

so this converges absolutely.

6. Find the radius and interval of convergence for the following power series:

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(x+3)^n}{\sqrt{n}}$ (b) $\displaystyle\sum_{n=1}^{\infty} \frac{\cos(n\pi)x^n}{3^n}$ (c) $\displaystyle\sum_{n=1}^{\infty} \frac{(2x-8)^n}{n!}$

Solution:

(a) From the ratio test:

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(x+3)^{n+1}}{(x+3)^n} \cdot \frac{\sqrt{n}}{\sqrt{n+1}} = |x+3|.$$

$|x+3| < 1$ for $-4 < x < -2$. From $x = -2$ we have $\displaystyle\sum_{n=1}^{\infty} \frac{1^n}{\sqrt{n}}$, which diverges by p-series. From $x = -4$ we have $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$, which converges by the alternating series test, but does not converge absolutely by p-series. So the interval of convergence is $[-4, -2)$ with the left end-point conditional and the interior absolute.

(b) Noting that $\cos(n\pi) = (-1)^n$, the ratio test:

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(x/3)^{n+1}}{(x/3)^n} = \frac{x}{3}.$$

$|x/3| < 1$ for $-3 < x < 3$, and the left and right endpoints give $\sum 1^n$ and $\sum(-1)^n$, respectively. These each diverge by the test for divergence, so the interval of convergence is $(-3, 3)$ and is absolute.

(c) From the ratio test:

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(2x-8)^{n+1}}{(2x-8)^n} \cdot \frac{n!}{(n+1)!} = \lim_{n\to\infty} \frac{|2x-8|}{n+1} = 0.$$

We conclude that we have absolute convergence for all x.

7. Consider the integral $\displaystyle\int_0^{\infty} \frac{xe^{-x}}{1-e^{-x}}\,dx$. While the integral is convergent, the corresponding anti-derivative can not be computed in closed form with elementary functions.

(a) Use the substitution $u = 1 - e^{-x}$ to rewrite the integral. You may have to solve for x in terms of u.

(b) Rewrite the resulting integrand in terms of series.

(c) Integrate to get a series that when evaluates, will give the correct definite integral.

Solution:

(a) The substitution suggested yields $I = \displaystyle\int_0^1 \frac{-\ln(1-u)}{u}\,du$.

(b) Using the series for $\ln(1-u)$ of $-\sum_{n=0}^{\infty} \frac{u^{n+1}}{n+1}$, we get: $I = \displaystyle\int_0^1 \sum_{n=0}^{\infty} \frac{u^{n+1}du}{u(n+1)} = \sum_{n=0}^{\infty} \int_0^1 \frac{u^n du}{(n+1)}$.

(c) Integrating and evaluating gives $I = \displaystyle\sum_{n=0}^{\infty} \frac{1}{(n+1)^2} = \sum_{n=1}^{\infty} \frac{1}{n^2}$.

8. The sum $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a fairly famous one (you may have seen it before...) The first proof of its value relies on Taylor series and the properties of polynomials. We can recreate Newton's proof.

(a) Write out the full Maclaurin series for $\frac{\sin(x)}{x}$ and state its interval of convergence.

(b) Your result in (a) is a polynomial. The roots of it are at the roots of $\sin(x)$. Factor this polynomial according to all of its roots, but write them as $(1 \pm \frac{x}{c})$ instead of as $(x \pm c)$.

(c) Look closely at the resulting product. The positive and negative roots have a very similar form. Simplify accordingly.

(d) If you were to multiply this out, what would the coefficient of x^2 be? It should match the coefficient in part (a), so set them equal and solve for the resulting series.

Solution:

(a) From the known series for $\sin(x)$, we have $\displaystyle\frac{\sin(x)}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)!}$.

This converges for all real x.

(b) The roots for $\sin(x)$ are $\pm n\pi$ for all non-zero integers n, so we should be able to factor the polynomial $\displaystyle\frac{\sin(x)}{x} = \left(1 - \frac{x}{\pi}\right)\left(1 + \frac{x}{\pi}\right)\left(1 - \frac{x}{2\pi}\right)\left(1 + \frac{x}{2\pi}\right)\cdots\left(1 - \frac{x}{n\pi}\right)\left(1 + \frac{x}{n\pi}\right)\cdots$. Note that it's not obvious that this factorization technique still works on infinite products; take on faith that it does for now (it was proven by Weierstrauss).

(c) $\displaystyle\frac{\sin(x)}{x} = \left(1 - \frac{x^2}{\pi^2}\right)\left(1 - \frac{x^2}{2^2\pi^2}\right)\left(1 - \frac{x^2}{3^2\pi^2}\right)\cdots\left(1 - \frac{x^2}{n^2\pi^2}\right)\cdots$.

(d) The coefficient of x^2 comes from fully multiplying this out, where for each term of the product we can only "choose" the x^2 term once, so we choose the "1" term from the other pieces of the product. We are left with a coefficient of x^2 of $\displaystyle\frac{-1}{\pi^2} - \frac{1}{2^2\pi^2} - \frac{1}{3^2\pi^2} - \cdots = \frac{-1}{\pi^2}\sum_{n=1}^{\infty} \frac{1}{n^2}$. But this value should equal $\frac{-1}{3!}$, the actual coefficient of the series in (a).

We conclude that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.