Questions for recitation 14 April 2021

1. Find the Taylor series for the following functions about the indicated center a:

(a)	$f(x) = e^{-2x}$	$a = 0$
(b)	$f(x) = x^2 - 2x + 4$	$a = b$
(c)	$f(x) = \frac{1}{x^2}$	$a = 1$
(d)	$f(x) = 2^x$	$a = 1$

2. Consider the following steps for estimating $\displaystyle\int_0^1 \frac{dx}{1+x^4}$.
   (a) Calculate the first 4 non-zero terms of the Maclaurin series for the integrand.
   (b) Determine the radius of convergence of this series.
   (c) Using the series, evaluate the integral.
   (d) How should we bound the error involved in this integral?

3. Find the Maclaurin series for $f(x) = \dfrac{x^2}{1+x^2}$. For what values does it converge?

4. Find the sum of the following series by starting with a similar Taylor series and performing any necessary transformations.
   (a) $x + x^3 + \dfrac{x^5}{2!} + \dfrac{x^7}{3!} + \ldots$
   (b) $2 - 3 \cdot 2x + 4 \cdot 3x^2 - 5 \cdot 4x^3 + \ldots$
   (c) $\dfrac{x^2}{2} + \dfrac{x^4}{4} + \dfrac{x^6}{6} + \dfrac{x^8}{8} + \ldots$
   (d) $1 - \dfrac{3x^2}{2!} + \dfrac{5x^4}{4!} - \dfrac{7x^6}{6!} + \ldots.$

5. Determine whether the following series converge or diverge:
   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^2+1}$
   (b) $\displaystyle\sum_{n=1}^{\infty} \ln(2n) - \sum_{n=1}^{\infty} \ln(4n+2)$
   (c) $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n}$

6. Find the radius and interval of convergence for the following power series:

(a) $\sum_{n=1}^{\infty} \frac{(x+3)^n}{\sqrt{n}}$ (b) $\sum_{n=1}^{\infty} \frac{\cos(n\pi)x^n}{3^n}$ (c) $\sum_{n=1}^{\infty} \frac{(2x-8)^n}{n!}$

7. Consider the integral $\int_0^{\infty} \frac{xe^{-x}}{1-e^{-x}}\,dx$. While the integral is convergent, the corresponding anti-derivative can not be computed in closed form with elementary functions.

   (a) Use the substitution $u = 1 - e^{-x}$ to rewrite the integral. You may have to solve for x in terms of u.

   (b) Rewrite the resulting integrand in terms of series.

   (c) Integrate to get a series that when evaluates, will give the correct definite integral.

8. The sum $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a fairly famous one (you may have seen it before...) The first proof of its value relies on Taylor series and the properties of polynomials. We can recreate Newton's proof.

   (a) Write out the full Maclaurin series for $\frac{\sin(x)}{x}$ and state its interval of convergence.

   (b) Your result in (a) is a polynomial. The roots of it are at the roots of $\sin(x)$. Factor this polynomial according to all of its roots, but write them as $(1 \pm \frac{x}{c})$ instead of as $(x \pm c)$.

   (c) Look closely at the resulting product. The positive and negative roots have a very similar form. Simplify accordingly.

   (d) If you were to multiply this out, what would the coefficient of x^2 be? It should match the coefficient in part (a), so set them equal and solve for the resulting series.