Test 1 Solution

Friday, February 5, 2021 6:59 AM

1. (32 pts) Consider the improper integral

$$I = \int_k^\infty \frac{1}{x^2 - x}\, dx$$

(a) First evaluate the indefinite integral

$$\int \frac{1}{x^2 - x}\, dx.$$

Express your answer in terms of a single logarithm.

$$\int \frac{1}{x^2-x}\,dx = \int \frac{1}{x(x-1)}\,dx$$

$$= \int \left(\frac{1}{x-1} - \frac{1}{x}\right)dx$$

$$= \ln|x-1| - \ln|x| + C$$

$$= \ln\left|\frac{x-1}{x}\right| + C$$

$$\frac{1}{x(x-1)} = \frac{A}{x} + \frac{B}{x-1}$$

By cover up rule:

$$\Rightarrow A = \frac{1}{0-1} = -1$$

$$B = \frac{1}{1} = 1$$

(b) Evaluate the improper integral $I$ for $k = 3$

At $k = 3$,

$$\int_k^\infty \frac{1}{x^2-x}\,dx = \int_3^\infty \frac{1}{x^2-x}\,dx$$

$$= \lim_{t\to\infty} \int_3^t \frac{1}{x^2-x}\,dx$$

$$= \lim_{t\to\infty} \ln\left|\frac{x-1}{x}\right|\Bigg|_3^t$$

$$= \lim_{t\to\infty} \ln\left|\frac{x-1}{x}\right|\Big|_3$$

$$= \lim_{t\to\infty} \left(\ln\left|\frac{t-1}{t}\right| - \ln\left|\frac{3-1}{3}\right|\right)$$

$$= \lim_{t\to\infty} \left(\ln\left|\frac{t-1}{t}\right| - \ln\left(\frac{2}{3}\right)\right)$$

$$= \lim_{t\to\infty} \ln\left|\frac{t-1}{t}\right| - \ln\left(\frac{2}{3}\right)$$

$$= \ln\left|\lim_{t\to\infty} \frac{t-1}{t}\right| - \ln\left(\frac{2}{3}\right)$$

$$\overset{L'H}{=} \ln\left|\lim_{t\to\infty} \frac{1}{1}\right| - \ln\left(\frac{2}{3}\right)$$

$$= \ln 1 - \ln\left(\frac{2}{3}\right)$$

$$= -\ln\left(\frac{2}{3}\right) = \ln\left(\frac{3}{2}\right)$$

(c) For which values of $k$ will the improper integral $I$ converge, and for which will it diverge? Justify your answer.

$$\int_k^\infty \frac{1}{x^2-x}\,dx = \lim_{t\to\infty} \int_k^t \frac{1}{x^2-x}\,dx$$

$$= \lim_{t\to\infty} \ln\left|\frac{x-1}{x}\right|\Big|_k^t$$

$$= \lim_{t\to\infty} \left( \ln\left|\frac{t-1}{t}\right| - \ln\left|\frac{k-1}{k}\right| \right)$$

$$= -\ln\left|\frac{k-1}{k}\right|$$

$$= \begin{cases} \infty & \text{if } k = 1 \text{ (and more generally } k<1) \\ -\ln\left|\frac{k-1}{k}\right| & \text{if } k>1 \end{cases}$$

Hence, the integral converges for $k>1$ and diverges for $k \le 1$.

2. (33 pts) Evaluate each integral:

(a)

$$\int \sqrt{x} \ln x \, dx$$

| | $u$ | $dv$ |
|---|---|---|
| $+$ | $\ln x$ | $x^{1/2}$ |
| $-$ | $\frac{1}{x}$ | $\frac{2}{3}x^{3/2}$ |

$$\int \sqrt{x} \ln x \, dx = uv - \int v \, du$$

$$= \frac{2}{3}x^{3/2}\ln x - \int \frac{2}{3}x^{3/2} \cdot x^{-1} \, dx$$

$$= \frac{2}{3}x^{3/2}\ln x - \frac{2}{3}\int x^{1/2} \, dx$$

$$= \frac{2}{3}x^{3/2}\ln x - \frac{2}{3} \cdot \frac{2}{3}x^{3/2} + C$$

$$= \frac{2}{3}x^{3/2}\ln x - \frac{4}{9}x^{3/2} + C$$

(b)

$$\int_0^{\frac{3\sqrt{2}}{2}} \frac{x^2}{(9-x^2)^{3/2}}\,dx$$

$$\int_0^{\frac{3\sqrt{2}}{2}} \frac{x^2}{(9-x^2)^{3/2}}\,dx = \int_0^{\frac{3\sqrt{2}}{2}} \frac{x^2}{\left(\sqrt{9-x^2}\right)^3}\,dx$$

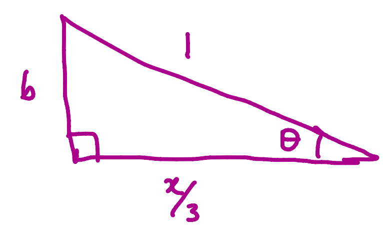
$$= \int_0^{\frac{3\sqrt{2}}{2}} \frac{x^2}{\left(\sqrt{9\left(1-\frac{x^2}{9}\right)}\right)^3}\,dx$$

$$= \int_0^{\frac{3\sqrt{2}}{2}} \frac{x^2}{\left(3\sqrt{1-\left(\frac{x}{3}\right)^2}\right)^3}\,dx$$

$$= \frac{1}{3^3}\int_0^{\frac{3\sqrt{2}}{2}} \frac{x^2}{\left(\sqrt{1-\left(\frac{x}{3}\right)^2}\right)^3}\,dx$$

Let $b = \sqrt{1-\left(\frac{x}{3}\right)^2}$. Then

$b^2 + \left(\frac{x}{3}\right)^2 = 1$




So take $\cos\theta = \frac{x}{3}$. Then $x = 3\cos\theta$ and

$-\sin\theta \frac{d\theta}{dx} = \frac{1}{3} \Rightarrow -3\sin\theta\, d\theta = dx.$

Thus,

$$\int_0^{\frac{3\sqrt{2}}{2}} \frac{x^2}{(9-x^2)^{3/2}}\,dx = \frac{1}{3^3}\int_0^{\frac{3\sqrt{2}}{2}} \frac{x^2}{\left(\sqrt{1-\left(\frac{x}{3}\right)^2}\right)^3}\,dx$$

$$= \frac{1}{27}\int_0^{\frac{3\sqrt{2}}{2}} \frac{9\cos^2\theta}{\left(\sqrt{1-\cos^2\theta}\right)^3} \cdot -3\sin\theta \, d\theta$$

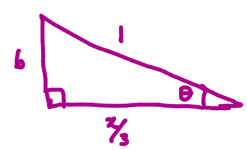
$$= -\int_0^{\frac{3\sqrt{2}}{2}} \frac{\cos^2\theta}{(\sin\theta)^3} \cdot \sin\theta \, d\theta$$

$$= -\int_0^{\frac{3\sqrt{2}}{2}} \frac{\cos^2\theta}{\sin^2\theta} \, d\theta$$

$$= -\int_0^{\frac{3\sqrt{2}}{2}} \frac{1-\sin^2\theta}{\sin^2\theta} \, d\theta$$

$$= -\int_0^{\frac{3\sqrt{2}}{2}} \left(\frac{1}{\sin^2\theta} - \frac{\sin^2\theta}{\sin^2\theta}\right) d\theta$$

$$= \int_0^{\frac{3\sqrt{2}}{2}} \left(-\csc^2\theta + 1\right) d\theta$$




$$= \left(\cot\theta + \theta\right)\Big|_{x=0}^{\frac{3\sqrt{2}}{2}}$$

$$\cot\theta = \frac{\text{Adj}}{\text{opp}} = \frac{x}{3} \div b$$

$$= \frac{x}{3\sqrt{1-\left(\frac{x}{3}\right)^2}}$$

$$= \frac{x}{\sqrt{9-x^2}}$$

$$= \frac{x}{\sqrt{9-x^2}} + \cos^{-1}\left(\frac{x}{3}\right)\Big|_0^{\frac{3\sqrt{2}}{2}}$$

$$= \left( \frac{\frac{3\sqrt{2}}{2}}{\sqrt{9 - \left(\frac{3\sqrt{2}}{2}\right)^2}} + \cos^{-1}\left(\frac{\frac{3\sqrt{2}}{2}}{3}\right)\right) - \left(0 + \cos^{-1}(0)\right)$$

$$= \frac{\frac{3\sqrt{2}}{2}}{\sqrt{9 - \frac{9\cdot 2}{4}}} + \cos^{-1}\left(\frac{\sqrt{2}}{2}\right) - \frac{\pi}{2}$$

since $0 \le x \le \frac{3\sqrt{2}}{2}$ and $\cos\theta = \frac{x}{3}$

$\Rightarrow 0 \le \cos\theta \le \frac{\sqrt{2}}{2}$

$\Rightarrow \frac{\pi}{4} \le \theta \le \frac{\pi}{2}.$

$$= \frac{\frac{3\sqrt{2}}{2}}{\sqrt{\frac{18}{4}}} + \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) - \frac{\pi}{2}$$

$$= \frac{\frac{3\sqrt{2}}{2}}{\frac{3\sqrt{2}}{2}} + \frac{\pi}{4} - \frac{\pi}{2}$$

$$= 1 - \frac{\pi}{4}$$

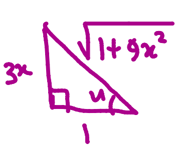
(c)

$$\int_0^{1/3} \tan^{-1}(3x)\,dx$$

(Careful: this is arctan, not tan)

| | u | dv |
|---|---|---|
| + | $\tan^{-1}(3x)$ | 1 |
| − | $\frac{3}{1+9x^2}$ | $x$ |

$u = \tan^{-1}(3x) \Rightarrow \tan u = 3x$




$\Rightarrow \sec^2 u \frac{du}{dx} = 3$

$\Rightarrow \frac{du}{dx} = 3\cos^2 u$

$= \frac{3}{1+9x^2}$

So,

$$\int_0^{1/3} \tan^{-1}(3x)\,dx = x\tan^{-1}(3x)\Big|_0^{1/3} - \int_0^{1/3} \frac{3x}{1+9x^2}\,dx$$

$$= \frac{1}{3}\tan^{-1}\left(\frac{3}{3}\right) - 0 - \frac{1}{6}\int_0^{1/3} \frac{6 \cdot 3x}{1+9x^2}\,dx$$

$$= \frac{1}{3}\tan^{-1}(1) - \frac{1}{6}\ln\left(1+9x^2\right)\Big|_0^{1/3}$$

$$= \frac{1}{3}\cdot\frac{\pi}{4} - \frac{1}{6}\left(\ln\left(1+9\left(\tfrac{1}{3}\right)^2\right) - \ln\left[1+9(0^2)\right]\right)$$

$$= \frac{\pi}{12} - \frac{1}{6}\left(\ln(1+1) - \ln(1)\right)$$

$$= \frac{\pi}{12} - \frac{1}{6}(\ln 2)$$

3. (24 pts) Determine whether each integral converges or diverges. Justify your answers completely.

(a)

$$\int_0^{10} \frac{1}{t^6+\sqrt{t}}\,dt$$

$$\int_0^{10} \frac{1}{t^6+\sqrt{t}}\,dt \leq \int_0^{10} \frac{1}{\sqrt{t}}\,dt$$

NB: $\frac{1}{2+3} < \frac{1}{3}$

$$= \int_0^{10} t^{-1/2}\, dt$$

$$= 2t^{1/2}\Big|_0^{10} = 2\sqrt{10}$$

Hence, $\int_0^{10} \frac{1}{t^6+\sqrt{t}}\, dt$ converges by comparison theorem

(b)

$$\int_1^{\infty} \frac{2x+1}{\sqrt{4x^3-1}}\, dx$$

$$\int_1^{\infty} \frac{2x+1}{\sqrt{4x^3-1}}\, dx \geq \int_1^{\infty} \frac{2x}{\sqrt{4x^3-1+1}}$$

NB: $\frac{1}{2} > \frac{1}{2+1}$

$1 - \frac{3}{2} = -\frac{1}{2}$

$$= \int_1^{\infty} \frac{2x}{\sqrt{4x^3}}$$

$$= \int_1^{\infty} \frac{1}{x^{1/2}}\, dx \text{ which diverges } \left(\text{Compare with } \int_1^{\infty} \frac{1}{x^p}\right)$$

Hence the $\int_1^{\infty} \frac{2x+1}{\sqrt{4x^3-1}}\, dx$ diverges by comparison theorem.

4. (21 pts) Suppose $f(x)$ and $g(x)$ are continuous for all $-\infty < x < \infty$. Suppose $\int_1^\infty f(x)\,dx$ converges, and that both $\int_{-\infty}^1 f(x)\,dx$ and $\int_{-\infty}^1 g(x)\,dx$ diverge. For each integral below, state whether it is convergent, divergent, or if the information above is insufficient to determine whether it conerges or diverges. No justification necessary.

(a)

$$\int_{-\infty}^{\infty} f(x)\,dx$$

$$\int_{-\infty}^{\infty} f(x)dx = \underbrace{\int_{-\infty}^{1} f(x)dx}_{\text{diverges}} + \underbrace{\int_{1}^{\infty} f(x)dx}_{\text{converges}}$$

So $\int_{-\infty}^{\infty} f(x)dx$ diverges.

(b)

$$\int_1^{1,000} g(x)\,dx$$

$\int_1^{1000} g(x)\,dx$ converges since $g$ is continuous (Compare with $\int_a^b g(x)\,dx$ when $g$ is continuous — FTC)

(c)

$$\int_{-1,000}^{\infty} f(x)\,dx$$

$$\int_{-1000}^{\infty} f(x)dx = \underbrace{\int_{-1000}^{1} f(x)dx} + \underbrace{\int_{1}^{\infty} f(x)dx}$$

converges since $f$ is continuous    converges

So the integral converges

(d)

$$\int_{-\infty}^{1} (f(x) + g(x))\, dx$$

$$\int_{-\infty}^{1} [f(x)+g(x)]\,dx = \begin{cases} \underbrace{\int_{-\infty}^{1} f(x)\,dx}_{\text{diverges}} + \underbrace{\int_{-\infty}^{1} g(x)\,dx}_{\text{diverges}}, & \text{if } f(x) \neq -g(x) \\ 0, & \text{if } f(x) = -g(x) \end{cases}$$

Hence, there is not enough information to conclude.

(e)

$$\int_{1}^{\infty} (f(x) - 1)\, dx$$

$$\int_{1}^{\infty} [f(x)-1]\,dx = \underbrace{\int_{1}^{\infty} f(x)\,dx}_{\text{converges}} - \underbrace{\int_{1}^{\infty} 1\,dx}_{\text{diverges}}$$

So $\int_1^\infty [f(x)-1]\,dx$ diverges

So $\int_1^\infty [f(x) - 1]\,dx$ diverges

(f)

$$\int_{-\infty}^{-1,000} g(x)\,dx$$

$$\int_{-\infty}^{-1000} g(x)dx = \underbrace{\int_{-\infty}^{1} g(x)dx}_{\text{diverges}} - \underbrace{\int_{-1000}^{1} g(x)dx}_{\text{converges}}$$

So $\int_{-\infty}^{-1000} g(x)\,dx$ diverges