1. Determine whether each series is absolutely convergent, conditionally convergent, or divergent. Justify your answer completely.

   (a) $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{(2n)!}{2^n n! n}$

The series diverges by the divergence test since

$$a_n = (-1)^n \frac{(2n)!}{2^n n! n} \xrightarrow{n\to\infty} \pm\infty$$

because $\dfrac{(n/2)^n}{n^2} \longrightarrow \infty$ as $n \to \infty$ and so

$$\frac{(2n)!}{2^n n! n} = \frac{(2n)(2n-1)(2n-2)\cdots(2n-n+1)(2n-n)!}{2^n n! n}$$

$$= \frac{(2n)(2n-1)(2n-2)\cdots(n+1)\cancel{n!}}{2^n \cancel{n!}\, n}$$

So $(n-1)$ terms remaining at the numerator

$$= \frac{n^{n-1}(2)(2-\frac{1}{n})(2-\frac{2}{n})\cdots(1+\frac{1}{n})}{2^n n}$$

$$= \frac{n^n(2)(2-\frac{1}{n})(2-\frac{2}{n})\cdots(1+\frac{1}{n})}{2^n n^2}$$

$$= \frac{(n/2)^n}{n^2}(2)(2-\tfrac{1}{n})(2-\tfrac{2}{n})\cdots(1+\tfrac{1}{n})$$

$$\longrightarrow \infty \cdot 2 \cdot 2 \cdot 2 \cdots 2 \cdot 1 = \infty.$$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{\cos n\pi}{n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ which converges conditionally as an alternating series with $b_n = \frac{1}{n}$ "and does not converge absolutely by p-series."

(c) $\displaystyle\sum_{n=0}^{\infty} ne^{-n}$

Using the root test,

$$|a_n|^{1/n} = |ne^{-n}|^{1/n} = n^{1/n} e^{-1} \xrightarrow{n\to\infty} e^{-1} = \frac{1}{e} < 1$$

So $\displaystyle\sum_{n=0}^{\infty} ne^{-n}$ converges absolutely by the root test.

NB: Integral test will also work.

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{e^{3n}}{n^n}$

Using the root test,

$$|a_n|^{1/n} = \left|\frac{e^{3n}}{n^n}\right|^{1/n} = \frac{e^3}{n} \xrightarrow{n\to\infty} 0 < 1$$

So $\displaystyle\sum_{n=1}^{\infty} \frac{e^{3n}}{n^n}$ converges absolutely.

(e) $\sum_{n=1}^{\infty} \frac{28n^2+5}{\sqrt{16n^7+3n^3+2}}$

$a_n = \frac{28n^2+5}{\sqrt{16n^7+3n^3+2}}$ is similar to $b_n = \frac{n^2}{n^{7/2}}$ as $n \to \infty$.

So using the limit comparison test,

$$\frac{a_n}{b_n} = \frac{28n^2+5}{\sqrt{16n^7+3n^3+2}} \div \frac{n^2}{n^{7/2}}$$

$$= \frac{28n^2+5}{\sqrt{16n^7+3n^3+2}} \times \frac{n^{7/2}}{n^2}$$

$$= \frac{28+\frac{5}{n^2}}{\sqrt{16+\frac{3}{n^4}+\frac{2}{n^7}}} \xrightarrow{n \to \infty} \frac{28}{\sqrt{16}} = 7 > 0$$

But $\sum_{n=1}^{\infty} \frac{n^2}{n^{7/2}} = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ converges by p-series.

Thus,

$\sum_{n=1}^{\infty} \frac{28n^2+5}{\sqrt{16n^7+3n^3+2}}$ converges by the Limit Comparison test.

(f) $-\frac{2}{2} + \frac{2\cdot 4}{2\cdot 5} - \frac{2\cdot 4\cdot 6}{2\cdot 5\cdot 8} + \ldots$

This is an alternating series with nth term

$$a_n = (-1)^n \frac{2\cdot 4\cdot 6 \cdots 2n}{}$$

$$a_n = (-1)^n \frac{2\cdot 4\cdot 6\cdot \cdots \cdot 2n}{2\cdot 5\cdot 8\cdot \cdots \cdot (3n-1)}$$

Using ratio test,

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{2\cdot 4\cdot 6\cdot \cdots \cdot 2n\,(2n+2)}{2\cdot 5\cdot 8\cdot \cdots \cdot (3n-1)(3n+2)} \div \frac{2\cdot 4\cdot 6\cdot \cdots \cdot 2n}{2\cdot 5\cdot 8\cdot \cdots \cdot (3n-1)}$$

$$= \frac{\cancel{2}\cdot \cancel{4}\cdot \cancel{6}\cdot \cdots \cdot \cancel{2n}\,(2n+2)}{\cancel{2}\cdot \cancel{5}\cdot \cancel{8}\cdot \cdots \cdot \cancel{(3n-1)}(3n+2)} \times \frac{\cancel{2}\cdot \cancel{5}\cdot \cancel{8}\cdot \cdots \cdot \cancel{(3n-1)}}{\cancel{2}\cdot \cancel{4}\cdot \cancel{6}\cdot \cdots \cdot \cancel{2n}}$$

$$= \frac{2n+2}{3n+2} \xrightarrow{n\to\infty} \frac{2}{3} < 1$$

Hence,

$-\frac{2}{2} + \frac{2\cdot 4}{2\cdot 5} - \frac{2\cdot 4\cdot 6}{2\cdot 5\cdot 8} + \cdots$ converges absolutely.

(g) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3+10\sqrt{n}}$

As $n\to\infty$, $a_n = \dfrac{1}{3+10\sqrt{n}}$ is similar to $b_n = \dfrac{1}{\sqrt{n}}$ with

$\displaystyle\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$ divergent by p-series

So by the limit comparison test,

$$\frac{a_n}{b_n} = \frac{1}{3+10\sqrt{n}} \div \frac{1}{\sqrt{n}}$$

$$= \frac{\sqrt{n}}{3+10\sqrt{n}} = \frac{1}{\frac{3}{\sqrt{n}}+10} \xrightarrow{n\to\infty} \frac{1}{10} > 0$$

Hence,

$$\sum_{n=1}^{\infty} \frac{1}{3+10\sqrt{n}}$$ diverges by the limit comparison test.

(h) $\displaystyle\sum_{n=1}^{\infty}(-1)^n \frac{3\cdot 9\cdot 15\cdots(6n-3)}{7^n n!}$

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{3\cdot 9\cdot 15\cdot\cdots\cdot(6n-3)(6n+3)}{7^{n+1}(n+1)!} \div \frac{3\cdot 9\cdot 15\cdot\cdots\cdot(6n-3)}{7^n n!}$$

$$= \frac{\cancel{3}\cdot\cancel{9}\cdot\cancel{15}\cdot\cdots\cdot\cancel{(6n-3)}(6n+3)}{\cancel{7^n}\cdot 7\cdot(n+1)\cdot\cancel{n!}} \times \frac{\cancel{7^n}\,\cancel{n!}}{\cancel{3}\cdot\cancel{9}\cdot\cancel{15}\cdot\cdots\cdot\cancel{(6n-3)}}$$

$$= \frac{6n+3}{7n+7} \xrightarrow{n\to\infty} \frac{6}{7} < 1$$

Hence,

$$\sum_{n=1}^{\infty}(-1)^n \frac{3\cdot 9\cdot 15\cdot\cdots\cdot(6n-3)}{7^n n!}$$

converges absolutely by the ratio test

(i) $\displaystyle\sum_{n=2}^{\infty}\frac{(-1)^n \ln n}{n^3}$

The series of absolute values $\displaystyle\sum_{n=2}^{\infty}\left|\frac{(-1)^n \ln n}{n^3}\right| = \sum_{n=2}^{\infty}\frac{\ln n}{n^3}$ is convergent

since by the integral test,

$\underbrace{\qquad\qquad}_{u \qquad dv}$

since by the integral test,

$$\int_2^\infty \frac{\ln x}{x^2}\,dx = -\frac{2}{x^3}\ln x\Big|_2^\infty + 2\int_2^\infty \frac{1}{x^4}\,dx$$

	u	dv
$+$	$\ln x$	$\frac{1}{x^2}$
$-$	$\frac{1}{x}$	$\frac{-2}{x^3}$

$$= \frac{2\ln 2}{2^3} + 2\left(\frac{1}{-3x^3}\right)\Big|_2^\infty$$

$\lim_{x\to\infty} \frac{-2\ln x}{x^3} = 0$ by L'Hospital's rule

$$= \frac{\ln 2}{4} + \frac{1}{4}.$$

Hence, $\sum_{n=2}^{\infty} \frac{(-1)^n \ln n}{n^3}$ is absolutely convergent.

(j) $\sum_{n=0}^{\infty} \frac{(3n)!}{(n!)^3}$

$$a_n = \frac{(3n)!}{(n!)^3} = \frac{(3n)!}{n!\,n!\,n!}$$

$$= \frac{\overbrace{3n(3n-1)(3n-2)(3n-3)\cdots(3n-(2n-1))}^{(2n-1)\text{ terms}}(3n-2n)!}{n!\,n!\,n!}$$

$$= \frac{\overbrace{3n(3n-1)(3n-2)(3n-3)\cdots(n+1)}^{(2n-1)\text{ terms}}\cancel{n!}}{n!\,n!\,\cancel{n!}}$$

$$\overbrace{3n(3n-1)(3n-2)(3n-3)\cdots(n+1)}^{(2n-1)\text{ terms}}$$

$$= \frac{3n(3n-1)(3n-2)(3n-3)\cdot\ldots\cdot(n+1)}{n!\,n!}$$

$$= \frac{n^{2n-1}\left(3-\frac{1}{n}\right)\left(3-\frac{2}{n}\right)\left(3-\frac{3}{n}\right)\cdot\ldots\cdot\left(1+\frac{1}{n}\right)}{n!\,n!}$$

$$= \frac{n^n\cdot n^n}{n}\cdot\frac{\left(3-\frac{1}{n}\right)\left(3-\frac{2}{n}\right)\left(3-\frac{3}{n}\right)\cdot\ldots\cdot\left(1+\frac{1}{n}\right)}{n!\,n!}$$

$$= \frac{n^n}{n\,n!}\cdot\frac{n^n}{n!}\cdot\left(3-\frac{1}{n}\right)\left(3-\frac{2}{n}\right)\left(3-\frac{3}{n}\right)\cdot\ldots\cdot\left(1+\frac{1}{n}\right)$$

$$\xrightarrow{n\to\infty} \infty\cdot\infty\cdot 3\cdot 3\cdot 3\cdot\ldots\cdot 1 = \infty$$

since

$$\frac{n^n}{n\,n!},\ \frac{n^n}{n!} \xrightarrow{n\to\infty} \infty$$

because

$$\frac{n^n}{n\,n!} = \frac{n\cdot n\cdot n\cdot n\cdot\ldots\cdot n}{n\cdot n(2n-1)(2n-2)\cdot\ldots\cdot 2\cdot 1}$$

$$= \frac{n}{2n-1}\cdot\frac{n}{2n-2}\cdot\frac{n}{2n-3}\cdot\ldots\cdot\frac{n}{3}\cdot\frac{n}{2} \geq \frac{n}{2} \quad \text{for all } n\geq 2$$

i.e.

$$\frac{n^n}{n!} \geq \frac{n^n}{n\,n!} \geq \frac{n}{2}$$

Hence, $\sum_{n=0}^{\infty}\frac{(3n)!}{(n!)^3}$ diverges by the divergence test.

Hence, $\sum_{n=0} \frac{(3n)!}{(n!)^3}$ diverges by the divergence test.

(k) $\sum_{n=2}^{\infty} \frac{n}{n^{5/2}-1}$

$a_n = \frac{n}{n^{5/2}-1}$ is similar to $b_n = \frac{n}{n^{5/2}}$ as $n \longrightarrow \infty$.

So using the limit comparison test,

$$\frac{a_n}{b_n} = \frac{n}{n^{5/2}-1} \div \frac{n}{n^{5/2}}$$

$$= \frac{n}{n^{5/2}-1} \times \frac{n^{5/2}}{n}$$

$$= \frac{n^{5/2}}{n^{5/2}-1}$$

$$= \frac{1}{1-\frac{1}{n^{5/2}}} \xrightarrow{n \to \infty} \frac{1}{1-0} = 1 > 0$$

But $\sum_{n=2}^{\infty} \frac{n}{n^{5/2}} = \sum_{n=2}^{\infty} \frac{1}{n^{1.5}}$ converges by p-series.

Hence, $\sum_{n=2}^{\infty} \frac{n}{n^{5/2}-1}$ converges absolutely by the limit comparison test.

(l) $\sum_{n=1}^{\infty} \frac{1}{n^{1/n}}$

$a_n = \frac{1}{\ldots} \xrightarrow{n \to \infty} \frac{1}{1} = 1 \neq 0$

$$a_n = \frac{1}{n^{1/n}} \xrightarrow{n \to \infty} \frac{1}{1} = 1 \neq 0$$

Hence, $\sum_{n=1}^{\infty} \frac{1}{n^{1/n}}$ diverges by the divergence test.

(m) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n3^n}$

Using the root test,

$$|a_n|^{1/n} = \left| \frac{(-1)^n}{n3^n} \right| = \frac{1}{n^{1/n} \cdot 3} \xrightarrow{n \to \infty} \frac{1}{3} < 1$$

Hence, $\sum_{n=1}^{\infty} \frac{(-1)^n}{n3^n}$ converges absolutely by the root test.

(n) $\sum_{n=1}^{\infty} \frac{\cos n\pi \sqrt[10]{n}}{n^{0.2}}$ $= \sum_{n=1}^{\infty} \frac{[\cos n\pi]\, n^{0.1}}{n^{0.2}}$

$= \sum_{n=1}^{\infty} \frac{(-1)^n}{n^{0.1}}$ which converges conditionally by alternating series test.

(o) $\sum_{n=1}^{\infty} \frac{(2\pi)^{3n}}{n^{2n}}$

Using the root test,

$$|a_n|^{1/n} = \left|\frac{(2\pi)^{3n}}{n^{2n}}\right|^{1/n} = \frac{(2\pi)^3}{n^2} \xrightarrow{n \to \infty} 0 < 1$$

Hence,

$\sum_{n=1}^{\infty} \frac{(2\pi)^{3n}}{n^{2n}}$ converges by the root test.

(p) $\sum_{n=1}^{\infty} \frac{6n-5}{n^3+n}$

$a_n = \frac{6n-5}{n^3+n}$ is similar to $b_n = \frac{n}{n^3} = \frac{1}{n^2}$ as $n \longrightarrow \infty$.

So using the limit comparison test,

$$\frac{a_n}{b_n} = \frac{6n-5}{n^3+n} \div \frac{1}{n^2}$$

$$= \frac{6n-5}{n^3+n} \times \frac{n^2}{1}$$

$$= \frac{6n^3-5n^2}{n^3+n}$$

$$= \frac{6 - \frac{5}{n}}{1 + \frac{1}{n^2}} \xrightarrow{n \to \infty} \frac{6}{1} = 6 > 0$$

Hence, $\sum_{n=1}^{\infty} \frac{6n-5}{n^3+n}$ converges by the limit comparison test since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by p-series.

(q) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(\arctan n)^n}$

Using the root test,

$$|a_n|^{1/n} = \left| \frac{1}{(\tan^{-1} n)^n} \right|^{1/n}$$

$$= \left| \frac{1}{\tan^{-1} n} \right| \xrightarrow{n \to \infty} \frac{1}{\pi/2} = \frac{2}{\pi} < 1$$

(r) $\displaystyle\sum_{n=1}^{\infty} \frac{5^{2n}}{n^2 9^n}$

Using the root test,

$$|a_n|^{1/n} = \left| \frac{5^{2n}}{n^2 9^n} \right|^{1/n}$$

$$= \frac{5^2}{n^{2/n} \cdot 9} \xrightarrow{n \to \infty} \frac{25}{9} > 1$$

Hence, $\displaystyle\sum_{n=1}^{\infty} \frac{5^{2n}}{n^2 9^n}$ diverges by the root test.

(s) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{2}{n} - e^{-n}\right) = \sum_{n=1}^{\infty} \frac{2}{n} - \underbrace{\sum_{n=1}^{\infty} \left(\frac{1}{e}\right)^n}_{\text{is series}}$

(s) $\sum_{n=1} \left(\frac{2}{n} - e^{-n}\right)$ $= \sum_{n=1} \frac{2}{n} - \underbrace{\sum_{n=1} (e)}_{\text{geometric series}}$

$$= \infty - \frac{1}{1-e^{-1}}$$

$$= \infty$$

Hence, $\sum_{n=1}^{\infty} \left(\frac{2}{n} - e^{-n}\right)$ diverges

2. Consider $g(x) = \dfrac{2}{3x+1}$

(a) Find a power series representation for $g(x)$ centered at 0

$$g(x) = \frac{2}{3x+1} = 2\left(\frac{1}{1-(-3x)}\right)$$

$$= 2\sum_{n=0}^{\infty} (-3x)^n, \qquad |-3x| < 1$$

$$= 2\sum_{n=0}^{\infty} (-1)^n 3^n x^n, \qquad |x| < \frac{1}{3}$$

(b) Use this result to find a power series representation for $-\dfrac{6x^3}{(3x+1)^2}$. What is its radius of convergence?

$$-\frac{6x^3}{(3x+1)^2} = x^3 \frac{d}{dx}\left(\frac{2}{3x+1}\right)$$

$$= x^3 \frac{d}{dx}\left(\sum_{n=0}^{\infty} (-1)^n 3^n x^n\right) \qquad |x| < \frac{1}{3} \qquad \text{from } ⓐ$$

$$= x^3 \sum_{n=1}^{\infty} (-1)^n 3^n \frac{d}{dx}(x^n), \qquad |x| < \frac{1}{3}$$

$$= x^3 \sum_{n=1}^{\infty} (-1)^n 3^n \frac{x^{n-1}}{n-1}, \qquad |x| < \frac{1}{3}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^n 3^n}{n-1} x^{n-1}, \qquad |x| < \frac{1}{3}$$

Radius of convergence is $\frac{1}{3}$.

3. (a) Use a Maclaurin series to find a series representation of sin(1/10)

The Maclaurin series of

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

So, the Maclaurin series of

$$\sin(1/10) = \sum_{n=0}^{\infty} (-1)^n \frac{(1/10)^{2n+1}}{(2n+1)!}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{10^{2n+1}(2n+1)!}$$

$$= \sum_{n=0} \frac{\quad}{10^{2n+1}(2n+1)!}$$

(b) How many terms of this series are needed to approximate sin(1/10) with an error less than 10^{-6}? (There are two estimation theorems relevant here: Taylor's Inequality and the Alternating Series Estimation Theorem. You will probably find it easier to use the latter.)

Since the series of $\sin(1/10)$ is an alternating series that satisfies the hypotheses of alternating series theorem, we use the Alternating Series Estimation Theorem.

$$|R_n| = |s - s_n| \leq b_{n+1}.$$

So, we are looking for n such that

$$b_{n+1} < 10^{-6}, \qquad b_n = \frac{1}{10^{2n+1}(2n+1)!}$$

$$\Rightarrow \quad \frac{1}{10^{2(n+1)+1}(2(n+1)+1)!} < 10^{-6}$$

$$\Rightarrow \quad \frac{1}{10^{2n+3}(2n+3)!} < 10^{-6}$$

$$\Rightarrow \quad \frac{1}{10^{2n+3}(2n+3)!} - 10^{-6} < 0, \quad n = ?$$

Using graphing calculator, we find that $n \geq 1$

Thus, at least two terms are need and

$$\sin(1/10) = \sum_{n=0}^{\infty} (-1)^n \frac{(1/10)^{2n+1}}{(2n+1)!} \approx \sum_{n=0}^{1} (-1)^n \frac{(1/10)^{2n+1}}{(2n+1)!}$$

$$= 0.998\bar{3}$$

is within 10^{-6} of the true sum.

(c) Find $\sum_{n=0}^{\infty}(-1)^n \frac{10\pi}{(2n+1)!}$ $= 10\pi \boxed{\sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)!}} \to \sin 1$

$$= 10\pi \sin(1)$$

4. Find the Taylor series, centered at 1, for $h(x) = 5/x^2$

The Taylor series centered at 1 is

$$h(x) = \sum_{n=0}^{\infty} \frac{h^{(n)}(1)}{n!}(x-1)^n.$$

But

$$h^{(0)}(1) = \left.\frac{5}{x^2}\right|_{x=1} = 5$$

$$h'(1) = \left.\frac{-5\cdot 2}{x^3}\right|_{x=1} = -5\cdot 2$$

$$h''(1) = \left.\frac{2\cdot 3\cdot 5}{x^4}\right|_{x=1} = 2\cdot 3\cdot 5$$

$$h'''(1) = \left.\frac{-2\cdot 3\cdot 4\cdot 5}{x^5}\right|_{x=1} = -2\cdot 3\cdot 4\cdot 5$$

$$h^{(4)}(1) = \left.\frac{2\cdot 3\cdot 4\cdot 5^2}{x^6}\right|_{x=1} = 2\cdot 3\cdot 4\cdot 5^2$$

$$h^{(5)}(1) = \left.\frac{-2\cdot 3\cdot 4\cdot 5^2\cdot 6}{x^7}\right|_{x=1} = -2\cdot 3\cdot 4\cdot 5^2\cdot 6$$

$$h^{(6)}(1) = \left.\frac{2\cdot 3\cdot 4\cdot 5^2\cdot 6\cdot 7}{x^8}\right|_{x=1} = 2\cdot 3\cdot 4\cdot 5^2\cdot 6\cdot 7$$

$$\vdots$$

$$h^{(n)}(1) = \left.(-1)^n\,\frac{2\cdot 3\cdot 4\cdot 5^2\cdot 6\cdot 7\cdot\ldots\cdot(n-1)}{x^{n+2}}\right|_{x=1} = (-1)^n\,5(n-1)!$$

Hence,

$$h(x) = \frac{5}{x^2} = \sum_{n=0}^{\infty} \frac{h^{(n)}(1)}{n!}(x-1)^n$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n 5(n-1)!}{n!}(x-1)^n$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n 5}{n}(x-1)^n$$

Furthermore,

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(-1)^{n+1}5\,(x-1)^{n+1}}{n+1} \div \frac{(-1)^n 5\,(x-1)^n}{n}\right|$$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\quad}{n+1} \div \frac{\quad}{n}\right|$$

$$= \left|\frac{5(x-1)^n(x-1)}{n+1} \times \frac{n}{5(x-1)^n}\right|$$

$$= \left|\frac{5(x-1)n}{n+1}\right| \longrightarrow 5|x-1| < 1$$

$\Rightarrow |x-1| < \frac{1}{5} \Rightarrow$ Radius of convergence is $\frac{1}{5}$.

Hence,

$$h(x) = \frac{5}{x^2} = \sum_{n=0}^{\infty} \frac{(-1)^n 5}{n}(x-1)^n \text{ on } \frac{4}{5} \leq x \leq \frac{6}{5}$$

since the series converges at $\frac{4}{5}$ and $\frac{6}{5}$.

5. Let $f(x) = x^{3/2}$

(a) Find the second degree Taylor polynomial $T_2(x)$ for $f(x)$, centered at 1

The second degree Taylor polynomial centered at 1 is

$$T_2(x) = \sum_{n=0}^{2} \frac{f^{(n)}(1)}{n!}(x-1)^n$$

$$= f(1) + f'(1)(x-1) + \frac{f''(1)}{2}(x-1)^2$$

$$= f(1) + f'(1)(x-1) + \frac{f''(1)}{2}(x-1)^2$$

But

$$f^{(0)}(1) = x^{3/2}\Big|_{x=1} = 1$$

$$f'(1) = \frac{3}{2}x^{1/2}\Big|_{x=1} = \frac{3}{2}$$

$$f''(1) = \frac{3}{2}\cdot\frac{1}{2}\cdot x^{-1/2}\Big|_{x=1} = \frac{3}{4}$$

Thus,

$$T_2(x) = 1 + \frac{3}{2}(x-1) + \frac{3}{4}(x-1)^2$$

(b) Use this result to approximate $\sqrt{(\frac{2}{3})^3}$. What is the maximum possible error in this approximation?

Taking $x = \frac{2}{3}$ in (a), we obtain:

$$T_2\left(\tfrac{2}{3}\right) = 1 + \frac{3}{2}\left(\tfrac{2}{3}-1\right) + \frac{3}{4}\left(\tfrac{2}{3}-1\right)^2$$

$$= 1 + \frac{3}{2}\left(-\tfrac{1}{3}\right) + \frac{3}{4}\left(-\tfrac{1}{3}\right)^2$$

$$= 1 - \frac{1}{2} + \frac{1}{12}$$

$$= \frac{12-6+1}{12}$$

$$= \frac{7}{12}$$

To find the maximum possible error in the approximation, we consider the interval $\left[1-\frac{2}{3}, 1+\frac{2}{3}\right] = \left[\frac{1}{3}, \frac{5}{3}\right]$ and use Taylor's Inequality as follows:

$$f^{(0)}(x) = x^{3/2}$$

$$f'(x) = \frac{3}{2} x^{1/2}$$

$$f''(x) = \frac{3}{2} \cdot \frac{1}{2} \cdot x^{-1/2}$$

$$f'''(x) = -\frac{3}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot x^{-3/2} = \frac{-3}{8x^{3/2}}$$

$$\Rightarrow M = \max_{x \in \left[\frac{1}{3}, \frac{5}{3}\right]} |f'''(x)| = \left| \frac{-3}{8x^{3/2}} \right| \Bigg|_{x=\frac{1}{3}} = \frac{3}{8\left(\frac{1}{3}\right)^{3/2}} = \frac{3\sqrt{27}}{8}$$

since $|f'''(x)|$ is decreasing on $\left[\frac{1}{3}, \frac{5}{3}\right]$.

Thus, by Taylor's Inequality,

$$|R_n(x)| \leq \frac{M|x-1|^{n+1}}{(n+1)!}$$

$$\Rightarrow |R_2(x)| \leq \frac{3\sqrt{27}\,\left|\frac{5}{3}-1\right|^3}{8(3!)} = \frac{\sqrt{27}}{2(27)} \approx 0.0962$$

Hence, the maximum possible error in this approximation is ≈ 0.0962 on the interval $\left[\frac{1}{3}, \frac{5}{3}\right]$.

on the interval $\left[\frac{1}{3}, \frac{5}{3}\right]$.

6. Suppose the power series $\sum_{n=0}^{\infty} c_n(x+1)^n$ is known to converge at $x = -6$ and diverge at $x = 4$. What is its radius of convergence? Its interval of convergence?

The power series is centered at $a = -1$ whose distance from 4 is 5. If the radius of convergence $R < 5$, then the power series would not be convergent at $x = -6$ (whose distance from -1 is also 5). If $R > 5$ then the power series would be convergent at $x = 4$.

Hence,

Radius of convergence $R = 5$

Interval of convergence $= [-1-5, -1+5) = [-6, 4)$.

7. Use series to evaluate $\lim_{x \to 0} \frac{3x^6 - x^8}{\arctan(x^2) - x^2}$

Using geometric series,

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n}, \quad |x| < 1$$

$$\Rightarrow \tan^{-1} x = \int_0^x \frac{1}{1+t^2}\, dt = \int_0^x \sum_{n=0}^{\infty} (-1)^n t^{2n}\, dt$$

$$\tan^{-1} = \sum^{\infty} (-1)^n \int^x t^{2n}\, dt$$

$$\Rightarrow \quad \tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \int_0^x t^{2n}\,dt$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

So

$$\tan^{-1}(x^2) = \sum_{n=0}^{\infty} (-1)^n \frac{(x^2)^{2n+1}}{2n+1}$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+2}}{2n+1}$$

$$= x^2 - \frac{x^6}{3} + \frac{x^{10}}{5} - \frac{x^{14}}{7} + \cdots$$

Thus,

$$\lim_{x\to 0} \frac{3x^6 - x^8}{\tan^{-1}(x^2) - x^2} = \lim_{x\to 0} \frac{3x^6 - x^8}{\left(x^2 - \frac{x^6}{3} + \frac{x^{10}}{5} - \frac{x^{14}}{7} + \cdots\right) - x^2}$$

$$= \lim_{x\to 0} \frac{3x^6 - x^8}{-\frac{x^6}{3} + \frac{x^{10}}{5} - \frac{x^{14}}{7} + \cdots}$$

$$= \lim_{x\to 0} \frac{x^6(3 - x^2)}{x^6\left(-\frac{1}{3} + \frac{x^4}{5} - \frac{x^8}{7} + \cdots\right)}$$

$$= \lim_{x \to 0} \frac{3 - x^2}{-\frac{1}{3} + \frac{x^4}{5} - \frac{x^8}{7} + \cdots}$$

$$= \frac{3 - 0^2}{-\frac{1}{3} + \frac{0^4}{5} - \frac{0^8}{7} + \cdots}$$

$$= \frac{3}{-1/3}$$

$$= \underline{\underline{-9}}$$

8. What is the sum of $\frac{1}{2^3 3!} - \frac{1}{2^5 5!} + \frac{1}{2^7 7!} - \frac{1}{2^9 9!} + \ldots$?

$$\frac{1}{2^3 3!} - \frac{1}{2^5 5!} + \frac{1}{2^7 7!} - \frac{1}{2^9 9!} + \cdots = \left(\frac{1}{2^1 1!}\right) - \left(\frac{1}{2^1 1!}\right) + \frac{1}{2^3 3!} - \frac{1}{2^5 5!} + \frac{1}{2^7 7!} - \frac{1}{2^9 9!} + \cdots$$

$$= \frac{1}{2} - \left(\frac{1}{2^1 1!} - \frac{1}{2^3 3!} + \frac{1}{2^5 5!} - \frac{1}{2^7 7!} + \frac{1}{2^9 9!} - \cdots\right)$$

$$= \frac{1}{2} - \sum_{n=0}^{\infty} (-1)^n \frac{(1/2)^{2n+1}}{(2n+1)!}$$

$$= \boxed{\frac{1}{2} - \sin(1/2)}$$

This looks simpler than the series but in practice, the series is simpler and easier to work with.

9. Find the radius and interval of convergence of $\sum_{n=0}^{\infty} \frac{\ln n (x+3)^n}{2^{n+1}}$ $= \sum_{n=0}^{\infty} \frac{(x+3)^n \ln n}{2^{n+1}}$

My interpretation, otherwise it won't be a power series.

Using ratio test,

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(x+3)^{n+1} \ln(n+1)}{2^{n+2}} \div \frac{(x+3)^n \ln n}{2^{n+1}}\right|$$

$$= \left|\frac{\cancel{(x+3)^n}(x+3)\ln(n+1)}{\cancel{2^n}\cdot \cancel{2^2}} \times \frac{\cancel{2^n}\cdot \cancel{2}}{\cancel{(x+3)^n}\ln n}\right|$$

$$= \left|\frac{(x+3)\ln(n+1)}{2\ln n}\right| \xrightarrow{n \to \infty} \frac{|x+3|}{2} < 1$$

$\Rightarrow |x+3| < 2 \iff -5 < x < -1$

$\Rightarrow$ Radius of convergence is 2.

At $x = -5$,

$$\sum_{n=1}^{\infty} \frac{(x+3)^n \ln n}{2^{n+1}} = \sum_{n=1}^{\infty} \frac{(-5+3)^n \ln n}{2^{n+1}}$$

$$= \sum_{n=1}^{\infty} \frac{(-2)^n \ln n}{2^{n+1}}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^n 2^n \ln n}{2^n \cdot 2}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^n \ln n}{2}$$ which diverges since $a_n \longrightarrow \pm\infty$

Similarly, at $x = -1$

$$\sum_{n=1}^{\infty} \frac{(x+3)^n \ln n}{2^{n+1}} = \sum_{n=1}^{\infty} \frac{(-1+3)^n \ln n}{2^{n+1}}$$

$$= \sum_{n=1}^{\infty} \frac{2^n \ln n}{2^n \cdot 2}$$

$$= \sum_{n=1}^{\infty} \frac{\ln n}{2}$$ which diverges since $a_n \longrightarrow \infty$.

Hence, interval of convergence is $(-5, -1)$.

10. Find the Maclaurin series for $g(x) = e^{-x^2}$. Use it to find the value of $g^{(20)}(0)$ (this should not require taking 20 derivatives).

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$ since $f(x) = e^x \Rightarrow f^{(n)}(0) = 1$.

$$\Rightarrow g(x) = e^{-x^2}$$

$$\Rightarrow \quad g(x) = e^{-x^2}$$

$$= \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!}$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{n!}$$

$$= 1 - \frac{x^2}{1!} + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^8}{4!} + \cdots - \frac{x^{18}}{9!} + \frac{x^{20}}{10!} - \cdots$$

Comparing with the general form $\sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} \frac{g^{(n)}(0)}{n!} x^n$, we get that

$$c_n = \frac{g^{(n)}(0)}{n!}$$

$$\Rightarrow \quad g^{(n)}(0) = c_n \, n!$$

So $\quad g^{(20)}(0) = \frac{1}{10!} \cdot 20!$

$$= \frac{20!}{10!}$$

11. (a) Find a power series representation of $\frac{-1}{(1+x)^2}$

$$\frac{-1}{(1+x)^2} = \frac{d}{dx}\left(\frac{1}{1+x}\right)$$

$(1+x)^{-}$

$$= \frac{d}{dx} \sum_{n=0}^{\infty} (-x)^n, \qquad |x|<1$$

$$= \sum_{n=1}^{\infty} (-1)^n \frac{d}{dx}(x^n)$$

$$= \sum_{n=1}^{\infty} (-1)^n n\, x^{n-1}, \qquad |x|<1$$

(b) Find a power series representation of $\frac{2}{(1+x)^3}$

$$\frac{2}{(1+x)^3} = \frac{d}{dx}\left[\frac{-1}{(1+x)^2}\right]$$

$$= \frac{d}{dx} \sum_{n=1}^{\infty} (-1)^n n\, x^{n-1}, \qquad |x|<1$$

from (a)

$$= \sum_{n=2}^{\infty} (-1)^n n \frac{d}{dx}(x^{n-1})$$

$$= \sum_{n=2}^{\infty} (-1)^n n(n-1)\, x^{n-2}, \qquad |x|<1$$

(c) Use these results to find the sum of $\sum_{n=2}^{\infty} \frac{n(n-1)}{2^{n-2}}$

Taking $x = -\frac{1}{2}$ in (b), we obtain

RHS: $$\sum_{n=2}^{\infty} (-1)^n n(n-1) \left(-\tfrac{1}{2}\right)^{n-2} = \sum_{n=2}^{\infty} (-1)^n n(n-1) (-1)^n \left(\frac{1}{2^{n-2}}\right)$$

$$= \sum_{n=2}^{\infty} (-1)^{2n} \frac{n(n-1)}{2^{n-2}}$$

$$= \sum_{n=2}^{\infty} \frac{n(n-1)}{2^{n-2}}.$$

LHS: $$\left.\frac{2}{(1+x)^3}\right|_{x=-1/2} = \frac{2}{\left(1-\frac{1}{2}\right)^3} = \frac{2}{(1/2)^3} = 2 \cdot 8 = 16.$$

Hence,

$$\sum_{n=2}^{\infty} \frac{n(n-1)}{2^{n-2}} = 16$$

12. Consider the sequence $\{a_n\}$, with $a_0 = 2$, and $a_{n+1} = \sqrt[3]{3 \cdot a_n}$. Find $\lim_{n\to\infty} a_n$

This problem is an application of the monotonic sequence Theorem

We need to show that $\lim_{n\to\infty} a_n$ makes sense by showing that $\{a_n\}$ is

(i) bounded

(ii) monotonic

(ii) monotonic

If so, then $\lim_{n\to\infty} a_n$ makes sense and in fact

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+1} \quad \text{(call this } x\text{)}$$

Thus,

$$a_{n+1} = \sqrt[3]{3a_n}$$

$$\Rightarrow \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \sqrt[3]{3a_n}$$

$$= \sqrt[3]{3 \lim_{n\to\infty} a_n}$$

$$\Rightarrow \quad x = \sqrt[3]{3x}$$

$$\Rightarrow \quad x^3 = 3x$$

$$\Rightarrow \quad x(x^2-3) = 0$$

$$\Rightarrow \quad x = 0, \; x = \pm\sqrt{3}.$$

Playing around with few terms of the sequence, we will notice that the right choice is $x = \sqrt{3}$.

Hence, we will need to show that $\{a_n\}$ is

(i) decreasing

(ii) bounded below by $\sqrt{3}$

so that

$$\lim_{n \to \infty} a_n = \sqrt{3}.$$

Both (i) and (ii) can be shown using the principle of mathematical induction.