The Comparison Tests-11.4: #29-32, 40-43, 45-46 and More Practice on
11.3: #22, 34, 41-44

Tuesday, March 23, 2021

(29) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n!}$

By definition,

$$n! = n(n-1)(n-2)(n-3)(n-4)\cdots 3\times 2\times 1$$
$$\geqslant 2\cdot 2\cdot 2\cdot 2\cdot 2 \cdots 2\times 2\times 1$$
$$= 2^{n-1}$$

$$\Rightarrow \quad \frac{1}{n!} \leq \frac{1}{2^{n-1}} \quad \Rightarrow \quad \sum_{n=1}^{\infty} \frac{1}{n!} \leq \sum_{n=1}^{\infty} \frac{1}{2^{n-1}}$$

And $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^{n-1}}$ Converges as a geometric series.

Hence, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n!}$ Converges by The Comparison Test.

(30) $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n}$

$$\frac{n!}{n^n} = \frac{n(n-1)(n-2)(n-3)(n-4)\cdots 3\times 2\times 1}{n\cdot n\cdot n\cdot n\cdot n\cdots n\cdot n\cdot n}$$

$$= \frac{(n-1)(n-2)(n-3)(n-4)\cdots 3\times 2\times 1}{n\cdot n\cdot n\cdot n\cdots n\cdot n\cdot n}$$

$$= \underbrace{\frac{n-1}{n}\cdot\frac{n-2}{n}\cdot\frac{n-3}{n}\cdot\frac{n-4}{n}\cdots\frac{3}{n}}_{\text{each of these } \leq 1}\cdot\frac{2\times 1}{n^2}$$

$$\leq \frac{2\times 1}{n^2}$$ assuming n is at least 2.

$\displaystyle\sum^{\infty} 2$

n^2

$$\Rightarrow \sum_{n=1}^{\infty} \frac{n!}{n^n} \leq \sum_{n=1}^{\infty} \frac{2}{n^2}.$$

Since $\sum_{n=1}^{\infty} \frac{2}{n^2}$ converges by p-series, we conclude that $\sum_{n=1}^{\infty} \frac{n!}{n^n}$ converges by The Comparison Test.

31) $\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$

Since $\lim_{x\to\infty} \frac{\sin\left(\frac{1}{x}\right)}{1/x} = \lim_{y\to 0} \frac{\sin y}{y} = 1$, then taking $a_n = \sin\left(\frac{1}{n}\right)$

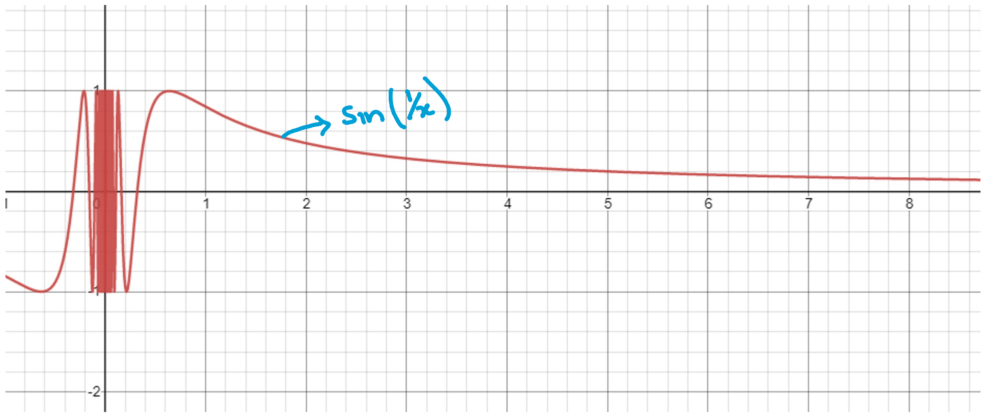
and $b_n = \frac{1}{n}$, we get:

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\sin(1/n)}{1/n} = 1.$$

But $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges by p-series.

Hence, $\sum_{n=1}^{\infty} \sin(1/n)$ diverges by The Limit Comparison Test.

Notice that $\sin\left(\frac{1}{n}\right)$ is positive as can be seen from the graph



(32) $\sum_{n=1}^{\infty} \frac{1}{n^{1+1/n}}$

The idea of solving this problem stems from the fact that

$\lim_{n\to\infty} n^{1/n} = 1$ and so $\lim_{n\to\infty} \frac{1}{n^{1/n}} = \frac{1}{\lim_{n\to\infty} n^{1/n}} = \frac{1}{1} = 1$

So taking $a_n = \frac{1}{n^{1+1/n}}$ and $b_n = \frac{1}{n}$, we get:

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1/n}{n^{1+1/n}} = \lim_{n\to\infty} \frac{1}{n^{1/n}} = \frac{1}{1} = 1.$$

But $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges by p-series.

Hence, $\sum_{n=1}^{\infty} \frac{1}{n^{1+1/n}}$ diverges by The Limit Comparison Test.

40 (a) $\sum a_n$, $\sum b_n$ have positive terms and $\sum b_n$ is convergent with $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$

By definition,

$\displaystyle\lim_{n\to\infty} \quad b_n$

By definition,

$\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = 0 \implies$ we can find $N > 1$ such that

$\displaystyle\left| \frac{a_n}{b_n} - 0 \right| < 1$ for all $n > N$.

$\displaystyle\implies \frac{a_n}{b_n} < 1$ for all $n > N$ since a_n, b_n are positive

$\implies a_n < b_n$ for all $n > N$

$$\implies \sum_{n=N+1}^{\infty} a_n \leq \sum_{n=N+1}^{\infty} b_n$$

$\implies \sum a_n$ converges by The Comparison Test since $\sum b_n$ converges

(b) (i) $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n^3}$

Taking $\displaystyle a_n = \frac{\ln n}{n^3}$ and $\displaystyle b_n = \frac{1}{n^2}$, we get:

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\ln n}{n} = \lim_{x\to\infty} \frac{\ln x}{x} \overset{L'H}{=} \lim_{x\to\infty} \frac{1/x}{1} = 0$$

So by part (a), we conclude that $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n^3}$ converges since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by p-series.

(b) (ii) $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{\sqrt{n}\, e^n}$

Taking $\displaystyle a_n = \frac{\ln n}{\sqrt{n}\, e^n}$ and $\displaystyle b_n = \frac{1}{e^n}$, we get:

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \left(\frac{\ln n}{\sqrt{n}\, e^n} \div \frac{1}{e^n} \right)$$

$$= \lim_{n\to\infty} \left(\frac{\ln n}{\sqrt{n}\, e^n} \times \frac{e^n}{1} \right)$$

$$= \lim_{n\to\infty} \frac{\ln n}{\sqrt{n}}$$

$$= \lim_{x\to\infty} \frac{\ln x}{\sqrt{x}}$$

$$\overset{L'H}{=} \lim_{x\to\infty} \frac{1/x}{\frac{1}{2}\cdot\frac{1}{\sqrt{x}}}$$

$$= \lim_{x\to\infty} \frac{2}{\sqrt{x}} = 0$$

So by part ⓐ, we conclude that $\sum_{n=1}^{\infty} \frac{\ln n}{\sqrt{n}\, e^n}$ converges since $\sum_{n=1}^{\infty} \frac{1}{e^n}$ converges as a geometric series.

41 ⓐ $\sum a_n$, $\sum b_n$ have positive terms and $\sum b_n$ is divergent with

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \infty.$$

By definition,

$\lim_{n\to\infty} \frac{a_n}{b_n} = \infty \Rightarrow$ we can find $N > 0$ such that

$\frac{a_n}{b_n} > 1$ for all $n > N$.

$$\Rightarrow \quad a_n > b_n \quad \text{for all } n > N.$$

$$\Rightarrow \quad \sum_{n=N+1}^{\infty} a_n \geq \sum_{n=N+1}^{\infty} b_n$$

So $\sum a_n$ diverges by The Comparison Test since $\sum b_n$ diverges.

(b)(i) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\ln n}$

Taking $a_n = \dfrac{1}{\ln n}$ and $b_n = \dfrac{1}{n}$, we get:

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n}{\ln n}$$

$$= \lim_{x\to\infty} \frac{x}{\ln x}$$

$$\overset{L'H}{=} \lim_{x\to\infty} \frac{1}{1/x} = \lim_{x\to\infty} x = \infty.$$

So by part (a), we conclude that $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\ln n}$ diverges since $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n}$ diverges by p-series.

(b)(ii) $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n}$

Taking $a_n = \dfrac{\ln n}{n}$ and $b_n = \dfrac{1}{n}$, we get

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \left(\frac{\ln n}{n} \div \frac{1}{n} \right)$$

$$= \lim_{n\to\infty} \left(\frac{\ln n}{n} \times \frac{n}{1} \right)$$

$$= \lim_{n\to\infty} \ln n = \infty$$

So by part ⓐ, we conclude that $\sum_{n=1}^{\infty} \frac{\ln n}{n}$ diverges since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges by p-series.

42 Take $a_n = \frac{1}{n^3}$ and $b_n = \frac{1}{n}$. Then

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \left(\frac{1}{n^3} \div \frac{1}{n} \right) = \lim_{n\to\infty} \frac{1}{n^2} = 0.$$

$\sum a_n$, $\sum b_n$ are positive but $\sum b_n$ diverges and $\sum a_n$ converges.

(This is a caution on the temptation to use 40 ⓐ wrongly. i.e., if $\sum b_n$ diverges, we cannot conclude 40 ⓐ is true)

43 $a_n > 0$ and $\lim_{n\to\infty} n a_n \neq 0$

$$\lim_{n\to\infty} n a_n = \lim_{n\to\infty} \frac{a_n}{1/n} > 0 \quad (\text{since } a_n > 0)$$

So $\sum a_n$ diverges by The Limit Comparison Test since $\sum \frac{1}{n}$ diverges by p-series.

(45) $\sum a_n$ is positive and convergent, $\sum \sin(a_n)$ convergent?

Just like in problem (31), $\sin(a_n)$ is positive if a_n is small enough or if n is large enough.

Moreover,

$$\lim_{n\to\infty} \frac{\sin(a_n)}{a_n} = 1.$$

So $\sum \sin(a_n)$ converges by The Limit Comparison Test since it is given that $\sum a_n$ converges.

(46) $\sum a_n$, $\sum b_n$ are positive and convergent; $\sum a_n b_n$ convergent?

Intuitively $\sum a_n$ and $\sum b_n$ are convergent

$\Rightarrow$ $\sum a_n b_n$ converges faster.

Now,

$\sum a_n$ converges $\Rightarrow$ $\lim\limits_{n\to\infty} a_n = 0$ $\Rightarrow$ we can find $N>0$ such that

$$|a_n - 0| < 1 \quad \text{for all } n>N.$$

$$\Rightarrow \quad a_n < 1 \quad \text{for all } n>N.$$

$$\Rightarrow \quad a_n b_n < b_n \quad \text{for all } n>N$$

$$\therefore \quad \sum^{\infty} a_n b_n \le \sum^{\infty} b_n$$

$$\Rightarrow \quad \sum_{n=N+1}^{\infty} a_n b_n \leq \sum_{n=N+1}^{\infty} b_n$$

$\Rightarrow \quad \sum a_n b_n$ converges by The Comparison Test since $\sum b_n$ converges (and recall that a finite number of terms missing do not affect convergent of a series)

22 $\displaystyle\sum_{n=2}^{\infty} \frac{\ln n}{n^2}$.

$f(x) = \dfrac{\ln x}{x^2}$ is positive, continuous and decreasing on $[2, \infty)$.

So

$$\int_2^{\infty} f(x)\,dx = \int_2^{\infty} \frac{\ln x}{x^2}\,dx$$

	u	dv
$+$	$\ln x$	x^{-2}
$-$	$1/x$	$-x^{-1}$

$$= \lim_{t\to\infty} \int_2^{t} \frac{\ln x}{x^2}\,dx$$

$$= \lim_{t\to\infty} \left(-\frac{\ln x}{x}\Big|_2^t + \int_2^t \frac{1}{x^2}\,dx \right)$$

$$= \lim_{t\to\infty} \left(\frac{-\ln t}{t} + \frac{\ln 2}{2} - \frac{1}{x}\Big|_2^t \right)$$

$$= \lim_{t\to\infty} \left(\frac{-\ln t}{t} + \frac{\ln 2}{2} - \frac{1}{t} + \frac{1}{2} \right)$$

$$= -\lim_{t\to\infty} \frac{\ln t}{t} - \lim_{t\to\infty} \frac{1}{t} + \frac{\ln 2}{2} + \frac{1}{2}$$

$$= -\lim_{t\to\infty} \frac{\ln t}{t} - 0 + \frac{\ln 2}{2} + \frac{1}{2}$$

$$\overset{L'H}{=} -\lim \frac{1}{t} + \frac{\ln 2}{2} + \frac{1}{2}$$

$$\overset{L'H}{=} \; -\lim_{t\to\infty} \frac{1/t}{1} + \frac{\ln 2}{2} + \frac{1}{2}$$

$$= \; -0 + \frac{\ln 2}{2} + \frac{1}{2}$$

$$= \; \frac{\ln 2}{2} + \frac{1}{2}.$$

$$\Longrightarrow \int_2^\infty \frac{\ln x}{x^2}\,dx \quad \text{converges}$$

Hence, $\sum_{n=2}^{\infty} \frac{\ln n}{n^2}$ converges by The Integral Test.

34 $\xi(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$

a $\sum_{n=2}^{\infty} \frac{1}{n^2} = \sum_{n=2}^{\infty} \frac{1}{n^2} + \frac{1}{1^1} - \frac{1}{1^1}$

$$= \left(\sum_{n=2}^{\infty} \frac{1}{n^2} + \frac{1}{1^1}\right) - \frac{1}{1^1}$$

$$= \sum_{n=1}^{\infty} \frac{1}{n^2} - 1$$

$$= \frac{\pi^2}{6} - 1$$

(b) $\displaystyle\sum_{n=3}^{\infty} \frac{1}{(n+1)^2}$

Let $m = n+1$. Then $n = m-1$.

So

$$\sum_{n=3}^{\infty} \frac{1}{(n+1)^2} = \sum_{m-1=3}^{\infty} \frac{1}{m^2}$$

$$= \sum_{m=4}^{\infty} \frac{1}{m^2}$$

$$= \sum_{m=4}^{\infty} \frac{1}{m^2} + \sum_{m=1}^{3} \frac{1}{m^2} - \sum_{m=1}^{3} \frac{1}{m^2}$$

$$= \left(\sum_{m=4}^{\infty} \frac{1}{m^2} + \sum_{m=1}^{3} \frac{1}{m^2} \right) - \sum_{m=1}^{3} \frac{1}{m^2}$$

$$= \sum_{m=1}^{\infty} \frac{1}{m^2} - \left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} \right)$$

$$= \frac{\pi^2}{6} - \left(1 + \frac{1}{4} + \frac{1}{9} \right)$$

$$= \frac{\pi^2}{6} - \left(\frac{36 + 9 + 4}{36} \right)$$

$$= \frac{\pi^2}{6} - \frac{49}{36}$$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(2n)^2} = \sum_{n=1}^{\infty} \frac{1}{2^2 \cdot n^2}$

$$= \frac{1}{4} \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$= \frac{1}{4}\left(\frac{\pi^2}{6}\right)$$

$$= \frac{\pi^6}{24}$$

I will post videos on 41 – 43 involving bound for error in the approximation $\displaystyle\sum_{n=1}^{\infty} a_n \approx \sum_{k=1}^{n} a_k$.