27 $\displaystyle\sum_{n=1}^{\infty} \frac{\cos(\pi n)}{\sqrt{n}}$. Since $\cos(\pi n)$ is negative for some n,

We cannot use the integral test to determine if the series is convergent because $f(x) = \frac{\cos(\pi x)}{\sqrt{x}}$ fails to satisfy the hypotheses of being positive and decreasing on $[1, \infty)$ which are required in the integral test.

28 $\displaystyle\sum_{n=1}^{\infty} \frac{\cos^2 n}{1+n^2}$.

Here, $f(x) = \frac{\cos^2 x}{1+n^2}$ is continuous and positive on $[1,\infty)$ but not decreasing since the presence of the cosine function will cause f to oscillate causing it to increase in some intervals and decrease on some other intervals. Hence, the integral test cannot used because one of its hypotheses is not satisfied.

32 $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n^p}$

$$\int_1^\infty \frac{\ln x}{x^p}\,dx = \lim_{t\to\infty} \int_1^t \frac{\ln x}{x^p}\,dx$$

$$= \lim_{t\to\infty} \int_0^{\ln t} \frac{u}{x^p}\cdot x\,du$$

$$= \lim_{t\to\infty} \int_0^{\ln t} \frac{u}{x^{p-1}}\,du$$

$u = \ln x \Rightarrow \frac{du}{dx} = \frac{1}{x}$

$\Rightarrow x\,du = dx$

Also,
$u = \ln x \Rightarrow x = e^u$

Moreover,
$x = 1 \Rightarrow u = \ln 1 = 0$ and
$x = t \Rightarrow u = \ln t$

$$= \lim_{t\to\infty} \int_0^{\ln t} \frac{u}{(e^u)^{p-1}}\,du$$

$$= \lim_{t\to\infty} \int_0^{\ln t} u e^{(1-p)u}\,du$$

$$\begin{array}{ccc} & u & dv \\ + & u & e^{(1-p)u} \\ - & 1 & \dfrac{e^{(1-p)u}}{1-p} \end{array}$$

$$= \lim_{t\to\infty} \left(\frac{u e^{(1-p)u}}{1-p} \Bigg|_0^{\ln t} - \frac{1}{1-p} \int_0^{\ln t} e^{(1-p)u}\,du \right)$$

$$= \lim_{t\to\infty} \left(\frac{1}{1-p} \left(\ln t\, e^{(1-p)\ln t} \right) - \frac{1}{1-p} \cdot \frac{e^{(1-p)u}}{1-p} \Bigg|_0^{\ln t} \right)$$

$$= \lim_{t\to\infty} \left(\frac{1}{1-p} \left(\ln t\, e^{\ln t^{1-p}} \right) - \frac{1}{(1-p)^2} \left(e^{(1-p)\ln t} \right) \right)$$

$$= \lim_{t\to\infty} \left(\frac{1}{1-p} \left(\ln t \left(t^{1-p} \right) \right) - \frac{1}{(1-p)^2} e^{\ln t^{1-p}} \right)$$

$$= \lim_{t\to\infty} \left(\frac{t^{1-p}}{1-p} \ln t - \frac{t^{1-p}}{(1-p)^2} \right) \qquad (*)$$

Clearly, if $p=1$, then (*) does not make sense, so we need to have handled this case first. But then $p=1$

$$\Rightarrow \int_1^\infty \frac{\ln x}{x^p}\,dx = \int_1^\infty \frac{\ln x}{x}\,dx$$

$$u = \ln x \Rightarrow \frac{du}{dx} = \frac{1}{x}$$

$$\Rightarrow x\,du = dx$$

$$= \int_{x=1}^{\infty} \frac{u}{x} \cdot x\,du$$

$$= \int_{x=1}^{\infty} u\,du$$

$$= \lim_{t\to\infty} \left. \frac{u^2}{2} \right|_1^{x=t}$$

$$= \lim_{t\to\infty} \left. \frac{(\ln x)^2}{2} \right|_1^t$$

$$= \lim_{t\to\infty} \frac{1}{2}\left((\ln t)^2 - (\ln 1)^2\right)$$

$$= \frac{1}{2}\lim_{t\to\infty} (\ln t)^2 \longrightarrow \infty \quad \text{as } t \to \infty.$$

$\Rightarrow$ the series does not converge for $p=1$ by the integral test.

Now, for $p \neq 1$,

$$\int_1^{\infty} \frac{\ln x}{x^p}\,dx = \lim_{t\to\infty} \left(\frac{t^{1-p}}{1-p} \ln t - \frac{t^{1-p}}{(1-p)^2} \right)$$

$$= \begin{cases} \infty & \text{if } 1-p>0 \\ 0 & \text{if } 1-p<0 \end{cases}$$

$$\{ \infty \quad \text{if} \quad p<1$$

$$= \begin{cases} \infty & \text{if} \quad p < 1 \\ 0 & \text{if} \quad p > 1 \end{cases}$$

Hence, the series converges for $p > 1$ and diverges for $p \leq 1$.

(33) $\zeta(x) = \sum_{n=1}^{\infty} \frac{1}{n^x}$.

Recall that the domain of a function is the set of points for which the function makes sense. For ζ to make sense, the series has to be convergent.

But by p-series,

$$\zeta(x) = \sum_{n=1}^{\infty} \frac{1}{n^x} = \begin{cases} \text{Convergents} & \text{if} \quad x > 1 \\ \text{diverges} & \text{if} \quad x \leq 1 \end{cases}$$

Hence,

$$\text{dom}(\zeta) = (1, \infty) = \{x \in \mathbb{R} : x > 1\}.$$

(35) $\zeta(4) = \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$ —— (**)

(a) $$\sum_{n=1}^{\infty} \left(\frac{3}{n}\right)^4 = \sum_{n=1}^{\infty} \frac{3^4}{n^4}$$

$$= 81 \sum_{n=1}^{\infty} \frac{1}{n^4}$$

$$= 81\left(\frac{\pi^4}{90}\right) \quad \text{by } (**)$$

$$= \frac{9\pi^4}{10}$$

ⓑ $\displaystyle\sum_{k=5}^{\infty} \frac{1}{(k-2)^4}$. (We need to transform this to look like (**))

Let $n = k-2$. Then $k = n+2$.

So
$$\sum_{k=5}^{\infty} \frac{1}{(k-2)^4} = \sum_{n+2=5}^{\infty} \frac{1}{n^4}$$

$$= \sum_{n=3}^{\infty} \frac{1}{n^4}$$

$$= \sum_{n=3}^{\infty} \frac{1}{n^4} + \left(\sum_{n=1}^{2} \frac{1}{n^4} - \sum_{n=1}^{2} \frac{1}{n^4}\right)$$

Add and subtract to adjust for the missing terms

$$= \left(\sum_{n=3}^{\infty} \frac{1}{n^4} + \sum_{n=1}^{2} \frac{1}{n^4}\right) - \sum_{n=1}^{2} \frac{1}{n^4}$$

$$= \sum_{n=1}^{\infty} \frac{1}{n^4} - \left(\frac{1}{1^4} + \frac{1}{2^4}\right)$$

$$= \frac{\pi^4}{90} - \left(1 + \frac{1}{16}\right) \quad \text{by } (**)$$

$$= \frac{\pi^4}{90} - \frac{17}{16}$$

(36) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^4}$. $\quad f(x) = \frac{1}{x^4}$ is continuous, positive and decreasing on $[1,\infty)$.

(a) $$S_{10} = \sum_{n=1}^{10} \frac{1}{n^4} = \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} + \frac{1}{6^4} + \frac{1}{7^4} + \frac{1}{8^4} + \frac{1}{9^4} + \frac{1}{10^4}$$

$$\simeq 1.0820366$$

$$R_{10} \le \int_{10}^{\infty} \frac{1}{x^4}\,dx = \lim_{t\to\infty} \int_{10}^{t} \frac{1}{x^4}\,dx$$

$$= \lim_{t\to\infty} \left.\frac{x^{-3}}{-3}\right|_{10}^{t}$$

$$= \lim_{t\to\infty} \left(\frac{-1}{3t^3} + \frac{1}{3(10^3)}\right)$$

In general $\displaystyle\int_n^{\infty} \frac{1}{x^4}\,dx = \frac{1}{3n^3}$ (useful for (b))

$$= \lim_{t\to\infty}\left(\frac{-1}{3t^3}\right) + \frac{1}{3000}$$

$$= \frac{1}{3000}$$

$$\approx 0.000\overline{3}$$

So the error in saying $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^4} = \sum_{n=1}^{10} \frac{1}{n^4}$ is at most $0.000\overline{3}$

(b) If $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^4} = s$, then by (3) in the book,

$$s_n + \int^{\infty} \frac{1}{\cdots}\,dx \le s \le s_n + \int^{\infty} \frac{1}{\cdots}\,dx$$

$$s_{10} + \int_{11}^{\infty} \frac{1}{x^4}\,dx \leq s \leq s_{10} + \int_{10}^{\infty} \frac{1}{x^4}\,dx$$

$$\Rightarrow \quad 1.0820366 + \frac{1}{3(11^3)} \leq s \leq 1.0820366 + \frac{1}{3(10^3)}$$

$$\Rightarrow \quad 1.082287 \leq s \leq 1.0823699$$

As argued in the book, approximate further by the average | midpoint:

$$s \approx \frac{1.082287 + 1.0823699}{2} = 1.0823285$$

with error

$$\leq \max\{s - 1.082287,\ 1.0823699 - s\} = 0.0000415$$

Hence,

$$\sum_{n=1}^{\infty} \frac{1}{n^4} \approx 1.0823285 \quad \text{with approximation error} \leq 0.0000415.$$

ⓓ s_n being within 0.00001 of $\sum_{n=1}^{\infty} \frac{1}{n^4}$ means that

$$s_n + \int_{n+1}^{\infty} \frac{1}{x^4}\,dx \leq \sum_{n=1}^{\infty} \frac{1}{n^4} \leq s_n + \int_{n}^{\infty} \frac{1}{x^4}\,dx$$

and

$$\max\left\{ \int_{n+1}^{\infty} \frac{1}{x^4}\,dx,\ \int_{n}^{\infty} \frac{1}{x^4}\,dx \right\} < 0.00001$$

$$\Rightarrow \quad \int_{n}^{\infty} \frac{1}{x^4}\,dx < 0.00001$$

$$\Rightarrow \quad \frac{1}{3(n^3)} < 0.00001$$

$$\Rightarrow \quad \frac{1}{3(0.00001)} < n^3$$

$$\Rightarrow \quad n > 32.1897949$$

Hence,

$n = 33$ suffices

(38) $\sum_{n=1}^{\infty} n e^{-2n}$ correct (accurate) to 4 decimal places.

It's enough to find n such that

$$\sum_{n=1}^{\infty} n e^{-2n} - S_n \leq 0.00005.$$

$\Leftrightarrow$ find n such that

$$S_n + \int_{n+1}^{\infty} x e^{-2x}\,dx \leq \sum_{n=1}^{\infty} n e^{-2n} \leq S_n + \int_{n}^{\infty} x e^{-2x}\,dx$$

$f(x) = xe^{-2x}$ satisfies all hypotheses of integral test

$\Leftrightarrow$ find n such that

$$\max\left\{\int_{n+1}^{\infty} x e^{-2x}\,dx,\ \int_{n}^{\infty} x e^{-2x}\,dx\right\} \leq 0.00005$$

$$\Rightarrow \quad \int^{\infty} x e^{-2x}\,dx \leq 0.00005$$

u $\qquad dv$

$-2x$

	u	dv
+	x	e^{-2x}
−	1	$\frac{e^{-2x}}{-2}$
+	0	$\frac{e^{-2x}}{4}$

$$\Rightarrow \quad \int_n^\infty x e^{-2x}\,dx \le 0.00005$$

$$\Rightarrow \lim_{t\to\infty} \left(-\frac{x e^{-2x}}{2} - \frac{e^{-2x}}{4}\right)\Bigg|_n^t \le 0.00005$$

$$\Rightarrow \lim_{t\to\infty} \left(-\frac{t e^{-2t}}{2} - \frac{e^{-2t}}{4} + \frac{n e^{-2n}}{2} + \frac{e^{-2n}}{4}\right) \le 0.00005$$

$$\Rightarrow \quad \frac{n e^{-2n}}{2} + \frac{e^{-2n}}{4} \le 0.00005$$

$$\Rightarrow \quad \left(\frac{n}{2} + \frac{1}{4}\right) e^{-2n} \le 0.00005$$

$$\left(\frac{2n+1}{4}\right) e^{-2n} \le 0.00005$$

By hook or crook, we get that the smallest positive integer that satisfies this inequality is 6.

Hence,

$$\sum_{n=1}^{\infty} n e^{-2n} \approx \sum_{n=1}^{6} n e^{-2n} = e^{-2} + 2e^{-4} + 3e^{-6} + 4e^{-8} + 5e^{-10} + 6e^{-12}$$

$$= 0.1810$$

and this is correct (accurate) to 4 decimal places.