Questions for recitation 19 March 2021

1. Exercises 11.3: #27-28, 32-33, 35-36, 38

2. Determine whether the following sums converge or diverge. Justify your answers. If you can figure out what a convergent sum converges to, do so.

(a) $\displaystyle\sum_{n=4}^{323} \frac{1}{\sqrt{n}}$

(b) $\displaystyle\sum_{n=3}^{\infty} \frac{\ln(n)}{n^2}$

(c) $\frac{1}{4} + \frac{1}{13} + \frac{1}{22} + \frac{1}{31} + \frac{1}{40} + \dots$

(d) $\displaystyle\sum_{n=19}^{\infty} \frac{18}{2+n^2}$

(e) $\displaystyle\sum_{n=-3}^{\infty} \cos\left(\frac{3}{n^2}\right).$

(f) $\displaystyle\sum_{n=1}^{\infty} \ln\left(\frac{1}{n}\right) - \ln\left(\frac{1}{n^2}\right)$

(g) $\displaystyle\sum_{n=9}^{\infty} 5(4)^{8-n}$

(h) $\displaystyle\sum_{n=10}^{\infty} \frac{1}{n\ln(n)}$

(i) $\displaystyle\sum_{n=4}^{\infty} \frac{n^2\sqrt{|\sin n\pi|}}{n^3+3}$

Solution:

(a) Converges. This is a finite sum.

(b) Converges by the integral test. $f(x) = \frac{\ln(x)}{x^2}$ is continuous, positive, decreasing for $x \geq 3$, and $\int_3^\infty f(x)\,dx \overset{u=\ln x}{=} \int_{\ln(3)}^\infty e^{-u}\,du \to 1/3$.

(c) This is $a_n = \frac{1}{9n-5}$, so $\sum a_n$ diverges by an integral test

(d) $\displaystyle\sum_{n=19}^{\infty} \frac{18}{2+n^2}$ converges by the integral test (via the limit $\tan^{-1}(x) \to \pi/2$)

(e) $\cos\left(\frac{3}{n^2}\right) \to 1$, so the sum diverges by the divergence test.

(f) $\ln\left(\frac{1}{n}\right) - \ln\left(\frac{1}{n^2}\right) = \ln\left(\frac{n^2}{n}\right) \to \infty$, so the sum diverges by the divergence test.

(g) $\displaystyle\sum_{n=9}^{\infty} 5(4)^{8-n} = \sum_{n=1}^{\infty} 5(4)^{-n} = \sum_{n=1}^{\infty} \frac{5}{4}\left(\frac{1}{4}\right)^{n-1} = \frac{\frac{5}{4}}{1-\frac{1}{4}} = \frac{5}{3}.$

(h) Diverges by the integral test: $f(x) = \frac{1}{x \ln x}$ is positive, decreasing, and continuous for $x \geq 10$, and

$$\int_{10}^{\infty} \frac{dx}{x \ln x} \overset{u=\ln x}{=} \lim_{t\to\infty} \ln(\ln x)|_{10}^{t} \to \infty$$

(i) Note that $\sin(n\pi) = 0$ for integer n. *Every* term of this series is exactly 0, so s_n is a constant sequence, and therefore converges to 0.

3. Find all non-negative values for α such that the infinite sum $\displaystyle\sum_{n} \alpha^{\ln(n)}$ converges.

Solution: Let's write out a few terms: $\displaystyle\sum_{n=1}^{\infty} \alpha^{\ln(n)} = \alpha^0 + \alpha^{\ln 2} + \alpha^{\ln 3} + \cdots$. This somewhat resembles a geometric-like growth. Using that as motivation, we can immediately see that the sum converges if $\alpha = 0$ (every term is zero) and the sum will diverge if $\alpha \geq 1$ (since the terms $a_n \nrightarrow 0$). $0 < \alpha < 1$ will require a little more work. Since $0 < \alpha < 1$, $f(x) = \alpha^{\ln x}$ is continuous, positive, and decreasing. We can use the integral test!

$$\begin{aligned}
\int_{1}^{\infty} \alpha^{\ln x}\, dx &\overset{u=\ln x}{=} \lim_{t\to\infty} \int_{0}^{t} \alpha^u (e^u du) \\
&= \lim_{t\to\infty} \int_{0}^{t} (\alpha e)^u \, du \\
&= \lim_{t\to\infty} \frac{(\alpha e)^u}{\ln(\alpha e)}\Big]_{0}^{t}
\end{aligned}$$

The limit tends to zero if $(\alpha e) < 1$ and to infinity if $(\alpha e) > 1$. When $(\alpha e) = 1$, the original series is the harmonic series and diverges. Putting this all together, we have that the sum converges for $0 \leq \alpha < \frac{1}{e}$.