Trigonometric Substitution

Friday, January 15, 2021

$\sin A \cos B = \frac{1}{2}\left(\sin(A-B) + \sin(A+B)\right)$

$\sin A \sin B = \frac{1}{2}\left(\cos(A-B) - \cos(A+B)\right)$

$\cos A \cos B = \frac{1}{2}\left(\cos(A-B) + \cos(A+B)\right)$

1. $\int \sin(4x)\cos(5x)\,dx$

Using the identity $\sin A \cos B = \frac{1}{2}\left(\sin(A-B) + \sin(A+B)\right)$,

$$\int \sin(4x)\cos(5x)\,dx = \int \frac{1}{2}\left(\sin(4x-5x) + \sin(4x+5x)\right)dx$$

$$= \int \frac{1}{2}\left(\sin(-x) + \sin(9x)\right)dx$$

$$= \frac{1}{2}\int(-\sin x)\,dx + \frac{1}{2}\int \sin(9x)\,dx$$

$$= \frac{1}{2}\cos x - \frac{1}{18}\cos(9x) + C$$

2. $\int \tan^3 x\,dx.$

$$\int \tan^3 x\,dx = \int \tan x \tan^2 x\,dx$$

$$= \int \tan x(\sec^2 x - 1)\,dx$$

$$= \int \tan x \sec^2 x\,dx - \int \tan x\,dx$$

$$= \int \tan x \sec^2 x\,dx - \ln|\sec x|$$

$u = \tan x \Rightarrow \frac{du}{dx} = \sec^2 x$

$\Rightarrow \frac{du}{\sec^2 x} = dx$

$$= \int u \cdot \sec^2 x \cdot \frac{du}{\sec^2 x} - \ln|\sec x|$$

$$= \int u\,du - \ln|\sec x|$$

$$= \frac{u^2}{2} - \ln|\sec x| + C$$

$$= \frac{\tan^2 x}{2} - \ln|\sec x| + C$$

3. Find $\int (\sin x + \cos x)^2\, dx$

$$\int (\sin x + \cos x)^2 dx = \int (\sin^2 x + 2\sin x \cos x + \cos^2 x) dx$$

$$= \int \sin^2 x\, dx + \int 2\sin x \cos x\, dx + \int \cos^2 x\, dx$$

$$= \frac{1}{2}\int (1 - \cos(2x)) dx + \int \sin(2x)\, dx + \frac{1}{2}\int (1 + \cos(2x)) dx$$

$$= \frac{1}{2}\left(x - \frac{1}{2}\sin(2x)\right) - \frac{1}{2}\cos(2x) + \frac{1}{2}\left(x + \frac{1}{2}\sin(2x)\right) + C$$

$$= \frac{1}{2}x - \frac{1}{2}\cos(2x) + \frac{1}{2}x + C$$

$$= x - \frac{1}{2}\cos(2x) + C$$

4. Find $\int \cos^4(2t)\, dt$

$$\int \cos^4(2t)\, dt = \int (\cos^2(2t))^2\, dt$$

$$\int \cos^4(2t)\,dt = \int \left(\cos^2(2t)\right)^2 dt$$

$$= \int \left(\tfrac{1}{2}\left(1+\cos(4t)\right)\right)^2 dt$$

$$= \tfrac{1}{4}\int \left(1+\cos(4t)\right)^2 dt$$

$$= \tfrac{1}{4}\int \left(1 + 2\cos(4t) + \cos^2(4t)\right) dt$$

$$= \tfrac{1}{4}\left(\int 1\,dt + 2\int \cos(4t)\,dt + \int \cos^2(4t)\,dt\right)$$

$$= \tfrac{1}{4}\left(t + \tfrac{2}{4}\sin(4t) + \tfrac{1}{2}\int \left(1+\cos(8t)\right)dt\right)$$

$$= \tfrac{1}{4}\left(t + \tfrac{1}{2}\sin(4t) + \tfrac{1}{2}t + \tfrac{1}{16}\sin(8t)\right) + C$$

$$= \tfrac{1}{4}\left(\tfrac{3t}{2} + \tfrac{1}{2}\sin(4t) + \tfrac{1}{16}\sin(8t)\right) + C$$

5. Find $\int_{\pi}^{2\pi} \cos^3(\frac{\theta}{2})\sin^5(\frac{\theta}{2})\,d\theta$

$$\int_{\pi}^{2\pi} \cos^3\left(\tfrac{\theta}{2}\right)\sin^5\left(\tfrac{\theta}{2}\right)d\theta = \int_{\pi}^{2\pi} \cos\left(\tfrac{\theta}{2}\right)\cos^2\left(\tfrac{\theta}{2}\right)\sin^5\left(\tfrac{\theta}{2}\right)d\theta$$

$$= \int_{\pi}^{2\pi} \cos\left(\frac{\theta}{2}\right) \left(1 - \sin^2\left(\frac{\theta}{2}\right)\right) \sin^5\left(\frac{\theta}{2}\right) d\theta$$

$u = \sin\left(\frac{\theta}{2}\right)$

$\Rightarrow \frac{du}{d\theta} = \frac{1}{2}\cos\left(\frac{\theta}{2}\right)$

$\Rightarrow \frac{2\,du}{\cos\left(\frac{\theta}{2}\right)} = d\theta$

$$= \int_{\pi}^{2\pi} \cos\left(\frac{\theta}{2}\right) (1 - u^2)\, u^5 \cdot \frac{2\,du}{\cos\left(\frac{\theta}{2}\right)}$$

$$= 2\int_{\pi}^{2\pi} (1-u^2)\, u^5\, du$$

$$= 2\int_{\pi}^{2\pi} (u^5 - u^7)\, du$$

$$= 2\left(\frac{u^6}{6} - \frac{u^8}{8}\right)\Bigg|_{\theta=\pi}^{\theta=2\pi}$$

$$= 2\left(\frac{\sin^6\left(\frac{\theta}{2}\right)}{6} - \frac{\sin^8\left(\frac{\theta}{2}\right)}{8}\right)\Bigg|_{\pi}^{2\pi}$$

$$= 2\left(\frac{\sin^6\left(\frac{2\pi}{2}\right)}{6} - \frac{\sin^8\left(\frac{2\pi}{2}\right)}{8}\right) - 2\left(\frac{\sin^6\left(\frac{\pi}{2}\right)}{6} - \frac{\sin^8\left(\frac{\pi}{2}\right)}{8}\right)$$

$$= 2\left(\frac{\sin^6\pi}{6} - \frac{\sin^8\pi}{8}\right) - 2\left(\frac{(1)^6}{6} - \frac{(1)^8}{8}\right)$$

$$= 0 - 2\left(\frac{1}{6} - \frac{1}{8}\right)$$

$$= -2\left(\frac{8-6}{48}\right) = -2\left(\frac{2}{48}\right) = -\frac{1}{12}$$

$$= -2\left(\frac{8-6}{48}\right) = -2\left(\frac{2}{48}\right) = -\frac{1}{12}$$

6. Find $\int (3 - 6\sin^2 x + 3\sin^4 x)\, dx$

$$\int (3 - 6\sin^2 x + 3\sin^4 x)\,dx$$

$$= \int \left(3 - 6 \cdot \tfrac{1}{2}(1-\cos(2x)) + 3(\sin^2 x)^2\right) dx$$

$$= \int \left(3 - 3(1-\cos(2x)) + 3\left(\tfrac{1}{2}(1-\cos(2x))\right)^2\right) dx$$

$$= \int \left(3 - 3 + 3\cos(2x) + 3\left(\tfrac{1}{4}\left(1 - 2\cos(2x) + \cos^2(2x)\right)\right)\right) dx$$

$$= \int \left(3\cos(2x) + 3\left(\tfrac{1}{4}\ 1 - 2\cos(2x) + \tfrac{1}{2}(1+\cos(4x))\right)\right) dx$$

$$= \int \left(3\cos(2x) + 3\left(\tfrac{1}{4} - \tfrac{1}{2}\cos(2x) + \tfrac{1}{8} + \tfrac{1}{8}\cos(4x)\right)\right) dx$$

$$= \int \left(3\cos(2x) + \tfrac{3}{4} - \tfrac{3}{2}\cos(2x) + \tfrac{3}{8} + \tfrac{3}{8}\cos(4x)\right) dx$$

$$= \int \left(\tfrac{9}{8} + \tfrac{3}{2}\cos(2x) + \tfrac{3}{8}\cos(4x)\right) dx$$

$$= \tfrac{9}{8}x + \tfrac{3}{4}\sin(2x) + \tfrac{3}{32}\sin(4x) + C$$

7. Find $\int \frac{\sec^4(2t)}{\tan^9(2t)}\,dt$

$$\int \frac{\sec^4(2t)}{\tan^9(2t)}\,dt = \int \frac{\sec^2(2t)\sec^2(2t)}{\tan^9(2t)}\,dt$$

$$= \int \frac{\sec^2(2t)\left(1+\tan^2(2t)\right)}{\tan^9(2t)}\,dt$$

$u = \tan(2t)$

$\Rightarrow \frac{du}{dt} = 2\sec^2(2t)$

$\Rightarrow \frac{du}{2\sec^2(2t)} = dt$

$$= \int \frac{\sec^2(2t)(1+u^2)}{u^9} \cdot \frac{du}{2\sec^2(2t)}$$

$$= \int \frac{(1+u^2)}{2u^9}\,du$$

$$= \frac{1}{2}\int \left(u^{-9} + u^{-7}\right)du$$

$$= \frac{1}{2}\left(\frac{u^{-8}}{-8} + \frac{u^{-6}}{-6}\right) + C$$

$$= \frac{1}{2}\left(-\frac{1}{8u^8} - \frac{1}{6u^6}\right) + C$$

$$= -\frac{1}{2}\left(\frac{1}{8\tan(2t)} + \frac{1}{6\tan(2t)}\right) + C$$

$$= -\frac{1}{2}\left(\frac{1}{8}\cot(2t) + \frac{1}{6}\cot(2t)\right) + C$$

8. Find $\int \frac{3+\sin^3 x}{\cos^2 x}\, dx$

$$\int \frac{3+\sin^3 x}{\cos^2 x}\, dx = \int\left(\frac{3}{\cos^2 x} + \frac{\sin^3 x}{\cos^2 x}\right) dx$$

$$= \int\left(3\sec^2 x + \frac{\sin x \sin^2 x}{\cos^2 x}\right) dx$$

$$= \int 3\sec^2 x\, dx + \int \frac{\sin x\,(1-\cos^2 x)}{\cos^2 x}\, dx$$

$u = \cos x$

$\Rightarrow \frac{du}{dx} = -\sin x$

$\Rightarrow \frac{du}{-\sin x} = dx$

$$= 3\tan x + \int \frac{\sin x\,(1-u^2)}{u^2} \cdot \frac{du}{-\sin x}$$

$$= 3\tan x + \int \frac{(1-u)}{-u^2}\, du$$

$$= 3\tan x + \int\left(\frac{1}{u} - \frac{1}{u^2}\right) du$$

$$= 3\tan x + \int \frac{1}{u}\, du - \int u^{-2}\, du$$

$$= 3\tan x + \ln|u| - \frac{u^{-1}}{-1} + c$$

$$= 3\tan x + \ln|\cos x| + \frac{1}{\cos x} + C$$

$$= 3\tan x + \ln|\cos x| + \sec x + C$$

9. Find $\int \cos(8t)\sin(t)\,dt$, in as simple a form as possible

Using the identity $\sin A \cos B = \frac{1}{2}\left(\sin(A-B) + \sin(A+B)\right)$,

$$\int \cos(8t)\sin t\,dt = \int \frac{1}{2}\left(\sin(t-8t) + \sin(t+8t)\right)dt$$

$$= \frac{1}{2}\int\left(\sin(-7t) + \sin(9t)\right)dt$$

$$= \frac{1}{2}\int\left(-\sin(7t) + \sin(9t)\right)dt$$

$$= \frac{1}{2}\left(\frac{\cos(7t)}{7} - \frac{\cos(9t)}{9}\right) + C$$

10. Find $\int_0^{\pi/8} \sin(6x)\sin(2x)\,dx$

Using the identity $\sin A \sin B = \frac{1}{2}\left(\cos(A-B) - \cos(A+B)\right)$,

$$\int_0^{\pi/8} \sin(6x)\sin(2x)\,dx = \int_0^{\pi/8} \frac{1}{2}\left(\cos(6x-2x) - \cos(6x+2x)\right)dx$$

$$= \frac{1}{2}\int_0^{\pi/8} \left(\cos(4x) - \cos(8x)\right) dx$$

$$= \frac{1}{2}\left(\frac{\sin(4x)}{4} - \frac{\sin(8x)}{8} \right)\Bigg|_0^{\pi/8}$$

$$= \frac{1}{2}\left(\frac{\sin\left(4\cdot\frac{\pi}{8}\right)}{4} - \frac{\sin\left(8\cdot\frac{\pi}{8}\right)}{8} \right) - \frac{1}{2}\left(\frac{\sin(4\cdot 0)}{4} - \frac{\sin(8\cdot 0)}{8} \right)$$

$$= \frac{1}{2}\left(\frac{\sin(\pi/2)}{4} - \frac{\sin(\pi)}{8} \right) - \frac{1}{2}\left(\frac{0}{4} - \frac{0}{8} \right)$$

$$= \frac{1}{2}\left(\frac{1}{4} - 0 \right) - 0$$

$$= \frac{1}{8}$$