Trigonometric Substitution

Thursday, January 21, 2021

1. Match the following integrals to their solutions *and know how to derive those solutions*:

(a)	$\int \tan(u)\,du$	(1)	$\frac{a^u}{\ln(a)} + C$
(b)	$\int \frac{du}{\sqrt{a^2-u^2}}$	(2)	$\frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C$
(c)	$\int \frac{du}{a^2+u^2}$	(3)	$\ln\lvert\sin(u)\rvert + C$
(d)	$\int \cot(u)\,du$	(4)	$\frac{1}{a}\sec^{-1}\left(\frac{u}{a}\right) + C$
(e)	$\int a^u\,du$	(5)	$\sin^{-1}\left(\frac{u}{a}\right) + C$
(f)	$\int \frac{du}{u\sqrt{u^2-a^2}}$	(6)	$-\ln\lvert\cos(u)\rvert + C$

(a) $\int \tan u\,du = \int \frac{\sin u}{\cos u}\,du$

Let $x = \cos u$. Then $\frac{dx}{du} = -\sin u \Rightarrow \frac{dx}{-\sin u} = du$.

Thus,

$$\int \tan u\,du = \int \frac{\sin u}{\cos u}\,du$$

$$= \int \frac{\sin u}{x} \cdot \frac{dx}{-\sin u}$$

$$= -\int \frac{1}{x}\,dx$$

$$= -\ln|x| + C$$

$$= -\ln|\cos u| + C$$

So

(a) $\int \tan u\,du$ (6) $-\ln|\cos u| + C$

(b) $\int \frac{du}{\sqrt{a^2-u^2}} = \int \frac{1}{\sqrt{a^2\left(1-\frac{u^2}{a^2}\right)}}\,du$, $a>0$

Let $b = \sqrt{1-\left(\frac{u}{a}\right)^2}$. Then

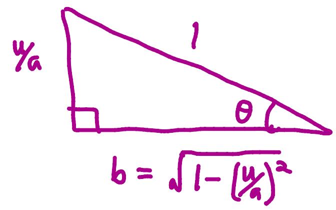
$b^2 + \left(\frac{u}{a}\right)^2 = 1$ (Pythagoras)

(b) $\int \frac{du}{\sqrt{a^2-u^2}} = \int \frac{du}{\sqrt{a^2\left(1-\frac{u^2}{a^2}\right)}}$

$$= \int \frac{1}{a\sqrt{1-\left(\frac{u}{a}\right)^2}}\,du$$

$$= \int \frac{1/a}{\sqrt{1-\left(\frac{u}{a}\right)^2}}\,du$$

$b^2 + \left(\frac{u}{a}\right)^2 = 1$ (Pythagoras)



So $\sin\theta = u/a \Rightarrow \cos\theta \frac{d\theta}{du} = \frac{1}{a}$

$$\Rightarrow \cos\theta \, d\theta = \frac{1}{a}du$$

$$\Rightarrow \sqrt{1-\left(\frac{u}{a}\right)^2}\, d\theta = \frac{1}{a}du$$

$$\Rightarrow d\theta = \frac{1/a}{\sqrt{1-\left(\frac{u}{a}\right)^2}}\,du$$

$$\cos\theta = \frac{\text{Adj}}{\text{Hyp}} = \frac{\sqrt{1-\left(\frac{u}{a}\right)^2}}{1}$$

Thus,

$$\int \frac{du}{\sqrt{a^2-u^2}} = \int \frac{1/a}{\sqrt{1-\left(\frac{u}{a}\right)^2}}\,du$$

$$= \int d\theta$$

$$= \theta + c$$

$$= \sin^{-1}\left(\frac{u}{a}\right) + c$$

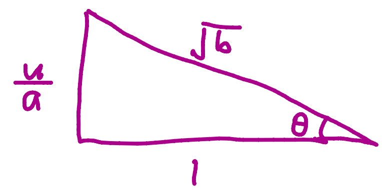
Hence

(b) $\displaystyle\int \frac{du}{\sqrt{a^2-u^2}}$ (5) $\sin^{-1}\left(\frac{u}{a}\right) + C$

(c) $$\int \frac{du}{a^2+u^2} = \int \frac{1}{a^2\left(1+\frac{u^2}{a^2}\right)}\,du$$

$$= \int \frac{1/a^2}{1+\left(\frac{u}{a}\right)^2}\,du$$

Let $b = 1+\left(\frac{u}{a}\right)^2$ (Think Pythagoreanly)



$\sec\theta = \frac{1}{\cos\theta} = \frac{1}{1/\sqrt{b}} = \sqrt{b}$

So $\tan\theta = \frac{u}{a} \Rightarrow \sec^2\theta \frac{d\theta}{du} = \frac{1}{a}$

$$\Rightarrow (\sec\theta)^2\, d\theta = \frac{1}{a}\,du$$

$$\Rightarrow b\, d\theta = \frac{1}{a}\,du$$

$$\Rightarrow d\theta = \frac{1/a}{b}\,du$$

$$= \frac{1/a}{\left(1+\left(\frac{u}{a}\right)^2\right)}\,du$$

$$\Rightarrow \frac{1}{a}\,d\theta = \frac{1/a^2}{\left(1+\left(\frac{u}{a}\right)^2\right)}\,du$$

Thus,

$$\int \frac{du}{a^2+u^2} = \int \frac{1/a^2}{\left(1+\left(\frac{u}{a}\right)^2\right)}\,du$$

$$= \int \frac{1}{a}\,d\theta$$

$$= \frac{1}{a}\theta + c$$

$$= \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + c$$

Hence,

ⓒ $\int \frac{du}{a^2+u^2}$ ② $\frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + c$

ⓓ $\int \cot u \, du = \int \frac{\cos u}{\sin u} du$

Let $x = \sin u$. Then $\frac{dx}{du} = \cos u$

$\Rightarrow \frac{dx}{\cos u} = du$

$$= \int \frac{\cos u}{x} \cdot \frac{dx}{\cos u}$$

$$= \int \frac{1}{x} dx$$

$$= \ln|x| + c$$

$$= \ln|\sin u| + c$$

Hence,

ⓓ $\int \cot u \, du$ ③ $\ln|\sin u| + c$

ⓔ $\int a^u du$

Let $x = a^u$. Then $\ln x = \ln a^u = u \ln a$

$\frac{d}{du}(\ln x) = \frac{d}{du}(u \ln a)$

Let $x = a^u$. Then $\ln x = \ln a^u = u\ln a$

$$\Rightarrow \frac{d}{du}(\ln x) = \frac{d}{du}(u\ln a)$$

$$\Rightarrow \frac{1}{x}\frac{dx}{du} = \ln a$$

$$\Rightarrow \frac{1}{x}\,dx = \ln a\,du$$

$$\Rightarrow \frac{1}{x\ln a}\,dx = du$$

Thus,

$$\int a^u du = \int x \cdot \frac{dx}{x\ln a}$$

$$= \int \frac{1}{\ln a}\,dx$$

$$= \frac{1}{\ln a}x + c$$

$$= \frac{1}{\ln a}a^u + c$$

Hence,

(a) $\int a^u du$ (i) $\dfrac{a^u}{\ln a} + c$

(f) $$\int \frac{du}{u\sqrt{u^2-a^2}} = \int \frac{1}{u\sqrt{a^2\left(\frac{u^2}{a^2}-1\right)}}\,du, \quad a>0$$

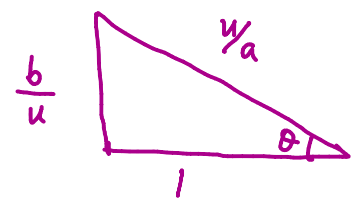
$$= \int \frac{1}{au\sqrt{\left(\frac{u}{a}\right)^2-1}}\,du$$

$$= \int \frac{1}{au\sqrt{\left(\frac{u}{a}\right)^2 - 1}}\, du$$

$$= \int \frac{1/a}{u\sqrt{\left(\frac{u}{a}\right)^2 - 1}}\, du$$

Let $b = u\sqrt{\left(\frac{u}{a}\right)^2 - 1}$. Then

$$\left(\frac{b}{u}\right)^2 = \left(\frac{u}{a}\right)^2 - 1 \Rightarrow \left(\frac{b}{u}\right)^2 + 1 = \left(\frac{u}{a}\right)^2$$



So $\cos\theta = \frac{1}{u/a} = \frac{a}{u}$ $\left(\text{ie; } \frac{u}{a} = \frac{1}{\cos\theta} = \sec\theta\right)$

$$\Rightarrow -\sin\theta \frac{d\theta}{du} = -\frac{a}{u^2}$$

$$\Rightarrow -\sin\theta\, d\theta = \frac{-a}{u^2}\, du$$

$$\Rightarrow d\theta = \frac{a}{u^2 \sin\theta}\, du$$

$$= \frac{a}{u^2\left(\frac{ab}{u^2}\right)}\, du$$

$$= \frac{1}{b}\, du$$

$$= \frac{1}{u\sqrt{\left(\frac{u}{a}\right)^2 - 1}}\, du$$

$$\sin\theta = \frac{\text{opp}}{\text{hyp}}$$

$$= \frac{b}{u} \div \frac{u}{a}$$

$$= \frac{b}{u} \times \frac{a}{u}$$

$$= \frac{ab}{u^2}$$

$$\Rightarrow \frac{d\theta}{a} = \frac{1/a}{u\sqrt{\left(\frac{u}{a}\right)^2 - 1}}\, du$$

$$\Rightarrow \quad \frac{d\theta}{a} = \frac{\cdot}{u\sqrt{\left(\frac{u}{a}\right)^2-1}}\, du$$

Thus,

$$\int \frac{du}{u\sqrt{u^2-a^2}} = \int \frac{1/a}{u\sqrt{\left(\frac{u}{a}\right)^2-1}}\, du$$

$$= \int \frac{1}{a}\, d\theta$$

$$= \frac{1}{a}\theta + c$$

$$= \frac{1}{a}\sec^{-1}\left(\frac{u}{a}\right) + c$$

Hence,

f $\int \frac{du}{u\sqrt{u^2-a^2}}$ 4 $\frac{1}{a}\sec^{-1}\left(\frac{u}{a}\right) + c$

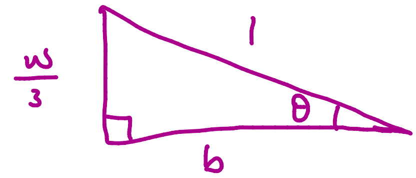
2. Evaluate $\int \frac{\sqrt{9-w^2}}{w^2}\, dw$

$$\int \frac{\sqrt{9-w^2}}{w^2}\, dw = \int \frac{\sqrt{9\left(1-\frac{w^2}{9}\right)}}{w^2}\, dw$$

$$= \int \frac{3\sqrt{1-\left(\frac{w}{3}\right)^2}}{w^2}\, dw$$

Let $b = \sqrt{1-\left(\frac{w}{3}\right)^2}$. Then

$$b^2 + \left(\frac{w}{3}\right)^2 = 1$$



So $\sin\theta = \frac{w}{3}$ $\left(\Rightarrow w^2 = 9\sin^2\theta\right)$

$$\Rightarrow \cos\theta \frac{d\theta}{dw} = \frac{1}{3}$$

$$\Rightarrow 3\cos\theta\, d\theta = dw$$

Thus,

$$\int \frac{\sqrt{9-w^2}}{w^2}\,dw = \int \frac{3\sqrt{1-\left(\frac{w}{3}\right)^2}}{w^2}\,dw$$

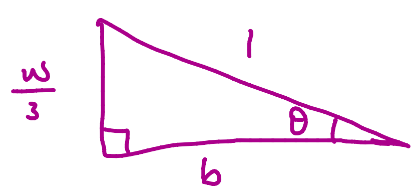
$$= \int \frac{3\sqrt{1-\sin^2\theta}}{9\sin^2\theta} \cdot 3\cos\theta\, d\theta$$

$$= \int \frac{3\cos\theta}{9\sin^2\theta} \cdot 3\cos\theta\, d\theta$$

$\sin^2\theta + \cos^2\theta = 1$

$\Rightarrow \cos\theta = \sqrt{1-\sin^2\theta}$

$$= \int \frac{\cos^2\theta}{\sin^2\theta}\,d\theta$$



$$= \int \frac{1-\sin^2\theta}{\sin^2\theta}\,d\theta$$

$\cot\theta = \frac{\text{Adj}}{\text{Opp}}$

$= \frac{b}{w/3}$

$= \sqrt{1-\left(\frac{w}{3}\right)^2}$

$$= \int \left(\frac{1}{\sin^2\theta} - \frac{\sin^2\theta}{\sin^2\theta}\right) d\theta$$

$$= \int (\operatorname{cosec}^2\theta - 1)\,d\theta \qquad = \frac{\sqrt{1-\left(\frac{w}{3}\right)^2}}{w/3}$$

$$= \int \operatorname{cosec}^2\theta - \int 1\,d\theta \qquad = \frac{3\sqrt{1-\left(\frac{w}{3}\right)^2}}{w}$$

$$= -\cot\theta - \theta + C \qquad \cot\theta = \frac{\sqrt{9-w^2}}{w}$$

$$= -\frac{\sqrt{9-w^2}}{w} - \sin^{-1}\left(\frac{w}{3}\right) + C$$

3. Evaluate the integral $\int \frac{t}{\sqrt{4t^2-1}}\,dt$:

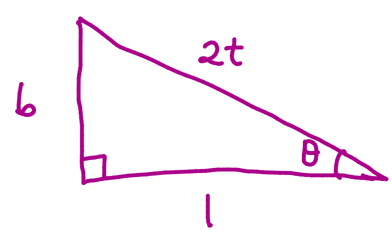
   (a) Using a trig substitution

   (b) Without using a trig substitution

   Are your answers the same?

Let $b = \sqrt{4t^2-1}$. Then

$$b^2 + 1 = (2t)^2$$



ⓐ $\int \frac{t}{\sqrt{4t^2-1}}\,dt$

So

$$\cos\theta = \frac{1}{2t} \quad \left(\Rightarrow t = \frac{1}{2\cos\theta} = \frac{1}{2}\sec\theta\right)$$

$\longrightarrow$ for fun, I used cosine - try sine, it works too!

$$\Rightarrow -\sin\theta \frac{d\theta}{dt} = -\frac{1}{2t^2}$$

$$\Rightarrow 2t^2 \sin\theta\, d\theta = dt$$

$$\Rightarrow \quad 2\left(\frac{1}{4\cos^2\theta}\right)\cdot \sin\theta \, d\theta = dt$$

$$\Rightarrow \quad \frac{\sin\theta}{\cos^2\theta}\, d\theta = dt$$

Thus,

$$\int \frac{t}{\sqrt{4t^2-1}}\, dt = \int \frac{\frac{1}{2}\sec\theta}{\sqrt{4\left(\left(\frac{1}{2}\sec\theta\right)^2-1\right)}} \cdot \frac{\sin\theta}{\cos^2\theta}\, d\theta$$

$$= \int \frac{\frac{1}{2}\sec\theta}{\sqrt{\sec^2\theta-1}} \cdot \frac{\sin\theta}{\cos\theta}\cdot\frac{1}{\cos\theta}\, d\theta$$

$$\frac{\cos^2\theta}{\cos^2\theta}+\frac{\sin^2\theta}{\cos^2\theta}=\frac{1}{\cos^2\theta}$$

$$= \int \frac{\frac{1}{2}\sec\theta}{\tan\theta} \cdot \tan\theta\sec\theta\, d\theta$$

$$\Rightarrow 1+\tan^2\theta = \sec^2\theta$$

$$\Rightarrow \tan^2\theta = \sec^2\theta - 1$$

$$\Rightarrow \tan\theta = \sqrt{\sec^2\theta-1}$$

$$= \int \frac{1}{2}\sec^2\theta\, d\theta$$

$$= \frac{1}{2}\tan\theta + C$$

$$= \frac{1}{2}\tan\left(\cos^{-1}\left(\frac{1}{2t}\right)\right) + C$$

6 $\int \frac{t}{\sqrt{4t^2-1}}\, dt$

(b) $\int \frac{t}{\sqrt{4t^2-1}}\,dt$

Let $u = 4t^2-1$. Then $\frac{du}{dt} = 8t \Rightarrow \frac{du}{8t} = dt$

Thus,

$$\int \frac{t}{\sqrt{4t^2-1}}\,dt = \int \frac{t}{u^{1/2}} \cdot \frac{du}{8t}$$

$$= \frac{1}{8}\int \frac{1}{u^{1/2}}\,du$$

$$= \frac{1}{8}\int u^{-1/2}\,du$$

$$= \frac{1}{8}\,\frac{u^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + c$$

$$= \frac{1}{8}\,\frac{u^{1/2}}{1/2} + c$$

$$= \frac{1}{4}\sqrt{4t^2-1} + c$$

Answers in (a) and (b) are not the same.

4. Evaluate $\int \frac{3x^2+1}{(x^2-9)^{3/2}}\,dx$. Simplify your answer as far as possible.

$$\int \frac{3x^2+1}{(x^2-9)^{3/2}}\,dx = \int \frac{3x^2+1}{\left(\sqrt{x^2-9}\right)^3}\,dx$$

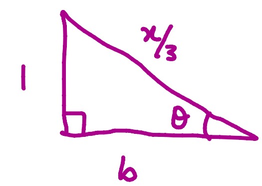
$$3x^2+1$$

$$= \int \frac{3x^2+1}{\left(\sqrt{9\left(\frac{x^2}{9}-1\right)}\right)^3}\,dx$$

$$= \int \frac{3x^2+1}{\left(3\sqrt{\left(\frac{x}{3}\right)^2-1}\right)^3}\,dx$$

$$= \frac{1}{3^3}\int \frac{3x^2+1}{\left(\sqrt{\left(\frac{x}{3}\right)^2-1}\right)^3}$$

Let $b = \sqrt{\left(\frac{x}{3}\right)^2-1}$. Then $b^2+1 = \left(\frac{x}{3}\right)^2$.



So, $\csc\theta = \frac{x}{3}$ $\left(\Rightarrow x = 3\csc\theta\right)$

$$\Rightarrow -\cot\theta\csc\theta\frac{d\theta}{dx} = \frac{1}{3} \Rightarrow -3\cot\theta\csc\theta\,d\theta = dx$$

Thus,

$$\int \frac{3x^2+1}{(x^2-9)^{3/2}}\,dx = \frac{1}{3^3}\int \frac{3x^2+1}{\left(\sqrt{\left(\frac{x}{3}\right)^2-1}\right)^3}$$

$$= \frac{1}{27}\int \frac{3(3\csc\theta)^2+1}{\left(\sqrt{\csc^2\theta-1}\right)^3}\cdot -3\cot\theta\csc\theta\,d\theta$$

$$\frac{\cos^2\theta+\sin^2\theta}{\sin^2\theta} = \frac{1}{\sin^2\theta}$$

$$\Rightarrow \cot^2\theta+1 = \csc^2\theta$$

$$\Rightarrow \cot = \sqrt{\csc^2\theta-1}$$

$$= -\frac{1}{9}\int \frac{27\csc^2\theta+1}{(\cot\theta)^3}\cdot\cot\theta\csc\theta\,d\theta$$

$$= -\frac{1}{9}\int \frac{27\csc^3\theta + \csc\theta}{\ldots}\,d\theta$$

$$= -\frac{1}{9}\int \frac{27\csc^3\theta + \csc\theta}{\cot^2\theta}\,d\theta$$

$$= -\frac{1}{9}\int \frac{27\csc^3\theta}{\cot^2\theta}\,d\theta - \frac{1}{9}\int \frac{\csc\theta}{\cot^2\theta}\,d\theta$$

$$= -3\int \frac{(1+\cot^2\theta)\csc\theta}{\cot^2\theta}\,d\theta - \frac{1}{9}\int \frac{1/\sin\theta}{\frac{\cos^2\theta}{\sin^2\theta}}\,d\theta$$

$$= -3\int \frac{\csc\theta}{\cot^2\theta}\,d\theta - 3\int \csc\theta\,d\theta - \frac{1}{9}\int \frac{\sin\theta}{\cos^2\theta}\,d\theta$$

$$= -3\int \frac{1/\sin\theta}{\frac{\cos^2\theta}{\sin^2\theta}}\,d\theta - 3\int \csc\theta\left(\frac{\cot\theta+\csc\theta}{\cot\theta+\csc\theta}\right)d\theta - \frac{1}{9}\int \frac{\sin\theta}{\cos^2\theta}\,d\theta$$

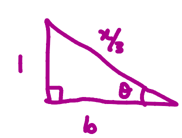
$$= -3\int \frac{\sin\theta}{\cos^2\theta}\,d\theta + 3\int \frac{-(\csc\theta\cot\theta + \csc^2\theta)}{\cot\theta+\csc\theta}\,d\theta - \frac{1}{9}\int \frac{\sin\theta}{\cos^2\theta}\,d\theta$$

$$= -\frac{28}{9}\int \frac{\sin\theta}{\cos^2\theta}\,d\theta - 3\ln|\cot\theta+\csc\theta|$$

$u = \cos\theta \Rightarrow \frac{du}{d\theta} = -\sin\theta$

$\Rightarrow \frac{du}{-\sin\theta} = d\theta$

$$= -\frac{28}{9}\int \frac{\sin\theta}{u^2}\cdot\frac{du}{-\sin\theta} - 3\ln|\cot\theta+\csc\theta|$$



$$= \frac{28}{9}\int \frac{1}{u^2}\,du - 3\ln|\cot\theta+\csc\theta|$$

$\sec\theta = \frac{Hyp}{Ad} = \frac{x}{3} \div b$

$$28\left(-\frac{1}{\;}\right) - 3\ln|\cot\theta+\csc\theta| + C$$

$= \frac{x}{3\sqrt{\left(\frac{x}{3}\right)^2 - 1}}$

$$= \frac{28}{9}\left(-\frac{1}{u}\right) - 3\ln\left|\cot\theta + \csc\theta\right| + C$$

$$= \frac{-1}{9\cos\theta} - 3\ln\left|\cot\theta + \csc\theta\right| + C$$

$$= -\frac{28}{9}\sec\theta - 3\ln\left|\cot\theta + \csc\theta\right| + C$$

$$= -\frac{28}{9}\left(\frac{x}{\sqrt{x^2-9}}\right) - 3\ln\left|\frac{1}{3}\sqrt{x^2-9} + \frac{x}{3}\right| + C$$

$$= \frac{x}{3\sqrt{\left(\frac{x}{3}\right)^2 - 1}}$$

$$= \frac{x}{\sqrt{x^2-9}}$$

$$\cot\theta = \frac{\text{Adj}}{\text{opp}} = \frac{b}{1}$$

$$= \sqrt{\left(\frac{x}{3}\right)^2 - 1} = \frac{1}{3}\sqrt{x^2-9}$$

and

$$\csc\theta = \frac{x}{3}$$

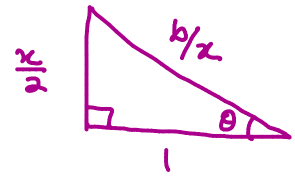
5. Evaluate $\int \frac{1}{x\sqrt{4x^2+16}}\,dx$

$$\int \frac{1}{x\sqrt{4x^2+16}}\,dx = \int \frac{1}{x\sqrt{16\left(\frac{1}{4}x^2+1\right)}}\,dx$$

$$= \int \frac{1}{4x\sqrt{\left(\frac{x}{2}\right)^2+1}}\,dx$$

Let $b = x\sqrt{\left(\frac{x}{2}\right)^2+1}$

$\Rightarrow \left(\frac{b}{x}\right)^2 = \left(\frac{x}{2}\right)^2 + 1$



So $\tan\theta = \frac{x}{2}$ $\left(\Rightarrow x = 2\tan\theta\right)$

$$\Rightarrow \sec^2\theta \frac{d\theta}{dx} = \frac{1}{2} \Rightarrow 2\sec^2\theta \, d\theta = dx$$

Thus,

$$\int \frac{1}{x\sqrt{4x^2+16}}\,dx = \int \frac{1}{4x\sqrt{\left(\frac{x}{2}\right)^2+1}}\,dx$$

$$= \int \frac{1}{4(2\tan\theta)\sqrt{\tan^2\theta+1}} \cdot 2\sec^2\theta \, d\theta$$

$$\frac{\sin^2\theta}{\cos^2\theta} + \frac{\cos^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta}$$

$$\Rightarrow \tan^2\theta + 1 = \sec^2\theta$$

$$\Rightarrow \sqrt{\tan^2\theta+1} = \sec\theta$$

$$= \int \frac{1}{8\tan\theta\sec\theta} \cdot 2\sec^2\theta \, d\theta$$

$$= \int \frac{\sec\theta}{4\tan\theta}\,d\theta$$

$$= \int \frac{1/\cos\theta}{4\left(\frac{\sin\theta}{\cos\theta}\right)}\,d\theta$$

$$= \frac{1}{4}\int \frac{1}{\sin\theta}\,d\theta$$

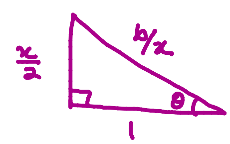
$$= \frac{1}{4} \int \operatorname{cosec}\theta \, d\theta$$

$$= \frac{1}{4} \int \operatorname{cosec}\theta \cdot \frac{\operatorname{cosec}\theta + \cot\theta}{\operatorname{cosec}\theta + \cot\theta} \, d\theta$$

$$= \frac{1}{4} \int \frac{\operatorname{cosec}^2\theta + \operatorname{cosec}\theta\cot\theta}{\operatorname{cosec}\theta + \cot\theta} \, d\theta$$

$$= -\frac{1}{4} \int \frac{-(\operatorname{cosec}^2\theta + \operatorname{cosec}\theta\cot\theta)}{\operatorname{cosec}\theta + \cot\theta} \, d\theta$$

$$= -\frac{1}{4} \ln \left| \operatorname{cosec}\theta + \cot\theta \right| + C$$



$$= -\frac{1}{4} \ln \left(\frac{\sqrt{x^2+4}}{x} + \frac{2}{x} \right) + C$$

$$\operatorname{cosec}\theta = \frac{\text{hyp}}{\text{opp}} = \frac{b}{x} \div \frac{x}{2} = \frac{2b}{x^2} = \frac{2x\sqrt{\left(\frac{x}{2}\right)^2+1}}{x^2} = \frac{\sqrt{x^2+4}}{x}$$

$$\cot\theta = \frac{\text{Adj}}{\text{opp}} = 1 \div \frac{x}{2} = \frac{2}{x}$$

6. Evaluate $\int_{-2}^{0} x(-x^2 - 4x)^{3/2} \, dx$. (Hint: Start with dealing with the "1/2" exponent by completing the square.)

$$\int_{-2}^{0} x(-x^2 - 4x)^{3/2} \, dx = \int_{-2}^{0} x\left(\sqrt{-x^2-4x}\right)^3 dx$$

By completing the square:

$$-x^2 - 4x = -(x^2+4x)$$

$$= -(x^2+4x+4-4)$$

$$-x^2-4x = -(x^2+4x)$$

$$= -(x^2+4x+4-4)$$

$$= -((x+2)^2-4)$$

$$= -(x+2)^2+4$$

$$= 4-(x+2)^2$$

$$= \int_{-2}^{0} x\left(\sqrt{4-(x+2)^2}\right)^3 dx$$

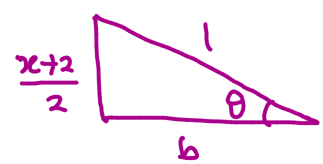
$$= \int_{-2}^{0} x\left(\sqrt{4\left(1-\left(\frac{x+2}{2}\right)^2\right)}\right)^3 dx$$

$$= \int_{-2}^{0} x\left(2\sqrt{1-\left(\frac{x+2}{2}\right)^2}\right)^3 dx$$

$$= 2^3\int_{-2}^{0} x\left(\sqrt{1-\left(\frac{x+2}{2}\right)^2}\right)^3 dx$$

Let $b = \sqrt{1-\left(\frac{x+2}{2}\right)^2}$. Then

$$b^2 + \left(\frac{x+2}{2}\right)^2 = 1$$



So $\sin\theta = \dfrac{x+2}{2}$ $\left(\begin{array}{l}\Rightarrow 2\sin\theta = x+2 \Rightarrow x = 2\sin\theta - 2 \\ \text{Also } x=-2 \Rightarrow \sin\theta = 0 \text{ and } x=0 \\ \Rightarrow \sin\theta = 1. \text{ So } 0 \le \sin\theta \le 1 \\ \Rightarrow \theta \in [0, \pi/2]\end{array}\right)$

$$\Rightarrow \cos\theta \frac{d\theta}{dx} = \frac{1}{2} \Rightarrow 2\cos\theta\, d\theta = dx$$

Thus,

$$\int_{-2}^{0} x(-x^2-4x)^{3/2} dx = 2^3\int_{-2}^{0} x\left(\sqrt{1-\left(\frac{x+2}{2}\right)^2}\right)^3 dx$$

$$= 8\int_{0}^{\pi/2} (2\sin\theta-2)\left(\sqrt{1-\sin^2\theta}\right)^3 \cdot 2\cos\theta\, d\theta$$

$$\cos^2\theta + \sin^2\theta = 1$$
$$\Rightarrow \cos^2\theta = 1-\sin^2\theta$$

$$= 32\int_{0}^{\pi/2} (\sin\theta-1)(\cos\theta)^3 \cos\theta\, d\theta$$

$$= 32\int_0^{\pi/2} (\sin\theta - 1)\,(\cos\theta)^3 \cos\theta\, d\theta$$

$\Rightarrow \cos^2\theta = 1-\sin^2\theta$

$\Rightarrow |\cos\theta| = \sqrt{1-\sin^2\theta}$

But $|\cos\theta| = \cos\theta$ on $[0, \frac{\pi}{2}]$

$$= 32\int_0^{\pi/2} \left(\sin\theta\cos^4\theta - \cos^4\theta\right) d\theta$$

$$= 32\int_0^{\pi/2} \sin\theta\cos^4\theta\, d\theta - 32\int_0^{\pi/2} \cos^4\theta\, d\theta$$

For $\int_0^{\pi/2} \sin\theta\cos^4\theta\, d\theta$, let $u = \cos\theta$. Then $\frac{du}{d\theta} = -\sin\theta \Rightarrow \frac{du}{-\sin\theta} = d\theta$

$$\Rightarrow \int_0^{\pi/2} \sin\theta\cos^4\theta\, d\theta = \int_0^{\pi/2} \sin\theta \cdot u^4 \cdot \frac{du}{-\sin\theta}$$

$$= -\int_0^{\pi/2} u^4\, du$$

$$= -\frac{u^5}{5}\Bigg|_{\theta=0}^{\theta=\pi/2}$$

$$= -\frac{\cos^5\theta}{5}\Bigg|_0^{\pi/2}$$

$$= -\frac{1}{5}\left(\cos^5\left(\frac{\pi}{2}\right) - \cos^5 0\right)$$

$$= -\frac{1}{5}(0-1) = \frac{1}{5}$$

Also,

$$\int_0^{\pi/2} \cos^4\theta\, d\theta = \int_0^{\pi/2} (\cos^2\theta)^2\, d\theta$$

$$\int_0^{\pi/2} \cos^4\theta \, d\theta = \int_0^{\pi/2} (\cos^2\theta)^2 d\theta$$

$$= \int_0^{\pi/2} \left(\tfrac{1}{2}(1+\cos(2\theta)\right)^2 d\theta$$

$$= \frac{1}{4}\int_0^{\pi/2} \left(1 + 2\cos(2\theta) + \cos^2(2\theta)\right) d\theta$$

$$= \frac{1}{4}\int_0^{\pi/2} \left(1 + 2\cos(2\theta) + \tfrac{1}{2}(1+\cos(4\theta))\right) d\theta$$

$$= \frac{1}{4}\int_0^{\pi/2} \left(\tfrac{3}{2} + 2\cos(2\theta) + \tfrac{1}{2}\cos(4\theta)\right) d\theta$$

$$= \frac{1}{4}\left(\tfrac{3}{2}\theta + \sin(2\theta) + \tfrac{1}{8}\sin(4\theta)\right)\Big|_0^{\pi/2}$$

$$= \frac{1}{4}\left(\tfrac{3}{2}\cdot\tfrac{\pi}{2} + \sin\pi + \tfrac{1}{8}\sin(2\pi)\right) - \frac{1}{4}\left(0 + \sin 0 + \tfrac{1}{8}\sin 0\right)$$

$$= \frac{1}{4}\left(\frac{3\pi}{4} + 0 + 0\right)$$

$$= \frac{3\pi}{16}$$

Thus,

$$\int_{-2}^{0} x(-x^2-4x)^{3/2} dx = 32\int_{-2}^{0} \sin\theta\cos^4\theta\, d\theta - 32\int_{-2}^{0} \cos^4\theta\, d\theta$$

$$= 32\left(\frac{1}{5}\right) - 32\left(\frac{3\pi}{16}\right)$$

$$= \frac{32}{5} - 6\pi$$