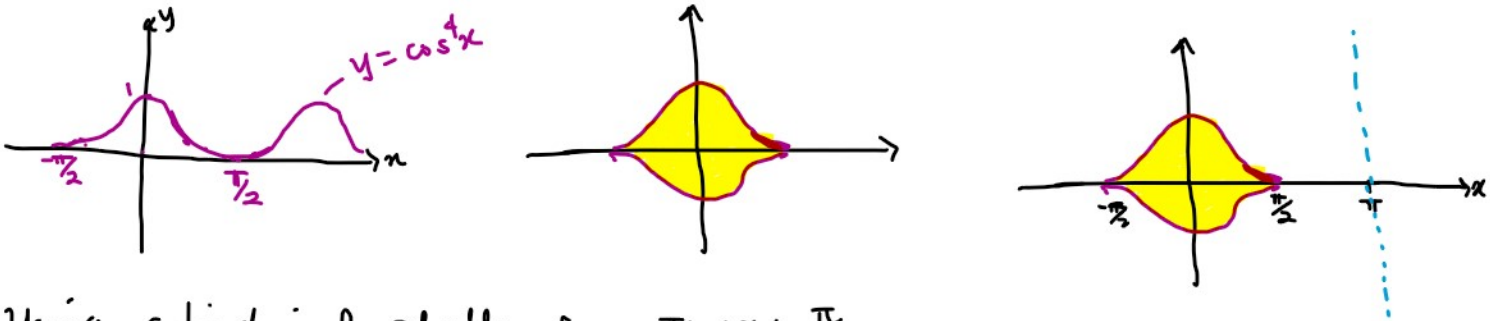
Volumes by Cylindrical Shells--6.3: #23, 24, 26, 29, 31, 44, 48

Friday, February 19, 2021

(23) $y = \cos^4 x,\ y = -\cos^4 x,\quad -\frac{\pi}{2} \le x \le \frac{\pi}{2}$; about $x = \pi$.



Using cylindrical shells, for $-\pi/2 \le x \le \pi/2$,

Radius $= \pi - x$

Circumference $= 2\pi(\pi - x)$

Height $= \cos^4 x - (-\cos^4 x) = 2\cos^4 x$

So, area of a shell

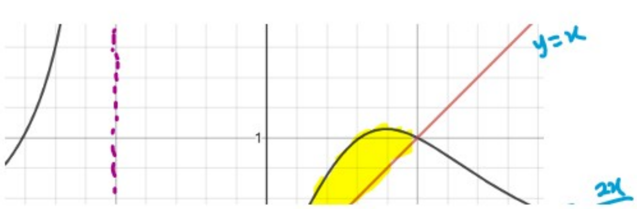
$$A(x) = 2\pi(\pi - x)(2\cos^4 x)$$
$$= 4\pi(\pi - x)\cos^4 x$$

Thus,

$$\text{Volume of solid} = \int_{-\pi/2}^{\pi/2} A(x)\,dx$$
$$= \int_{-\pi/2}^{\pi/2} 4\pi(\pi - x)\cos^4 x\,dx$$
$$\simeq \underline{\underline{46.50942}} \quad (\text{to } 5 \text{ d.p})$$

(24) $y = x,\ y = \dfrac{2x}{1+x^3}$; about $x = -1$.

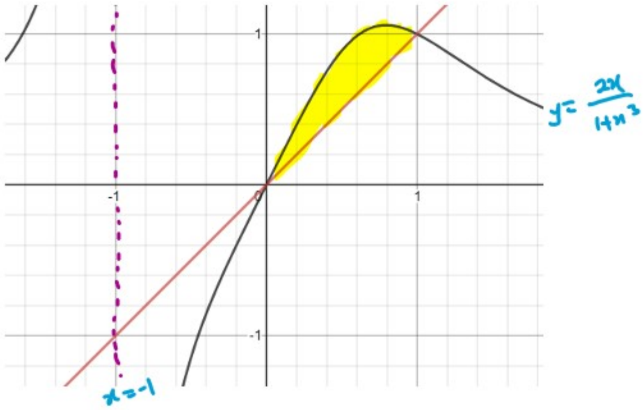
For any shell:



Radius $= x - (-1) = x + 1$

Circumference $= 2\pi(x+1)$

Height $= \dfrac{2x}{1+x^3} - x$



Circumference $= 2\pi(x+1)$

Height $= \frac{2x}{1+x^3} - x$

Since we are rotating about an axis parallel to y-axis and are using shells,

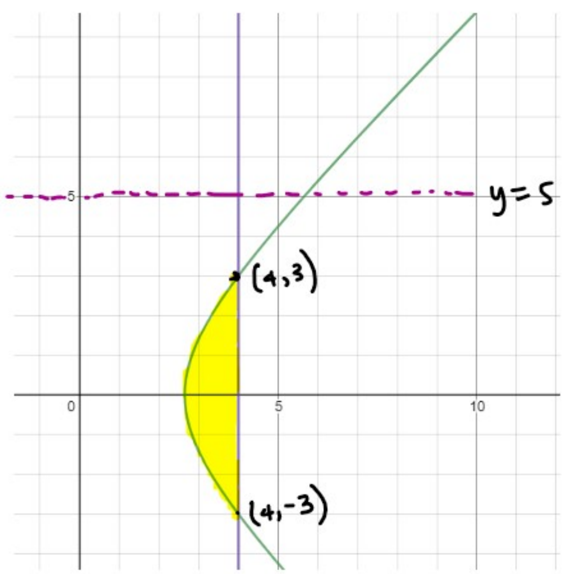
Volume of solid $= \int_0^1 A(x)\,dx$ But $A(x) = $ Circumference $\times$ height

$= 2\pi(x+1)\left(\frac{2x}{1+x^3} - x\right)$

$$= \int_0^1 2\pi(x+1)\left(\frac{2x}{1+x^3} - x\right)dx$$

$$= 2.36164$$

(26) $x^2 - y^2 = 7$, $x = 4$; about $y = 5$



For any shell:

Radius $= 5-y$

Circumference $= 2\pi(5-y)$

Height $= 4 - \sqrt{7+y^2}$ $\left(x^2-y^2=7 \Rightarrow x = \pm\sqrt{7+y^2}\right)$

Since axis of rotation is parallel to the x-axis and we are using the shells method,

So $A(y) = 2\pi(5-y)\left(4-\sqrt{7+y^2}\right)$

Since axis of rotation is parall to the x-axis and we ...

Volume of solid $= \int_{-3}^{3} A(y)\,dy$ But $A(y) = 2\pi(5-y)\left(4-\sqrt{7+y^2}\right)$

$$= \int_{-3}^{3} 2\pi(5-y)\left(4-\sqrt{7+y^2}\right)dy$$

$$= 163.02712$$

(29) $\int_0^3 2\pi x^5\,dx = \int_0^3 (2\pi x)\,x^4\,dx$ So Radius $= x$ $(x-0)$

Circumference $= 2\pi x$

Height $= x^4$ (x^4-0)

Thus, comparing with the form of volume using cylindrical shells, the solid is gotten by rotating the region bounded by $y = x^4$, $y=0$, $x=0$, $x=3$ about y-axis

(31) $2\pi\int_1^4 \frac{y+2}{y^2}\,dy = \int_1^4 2\pi(y+2)\left(\frac{1}{y^2}\right)dy$ So

Radius $= y+2$

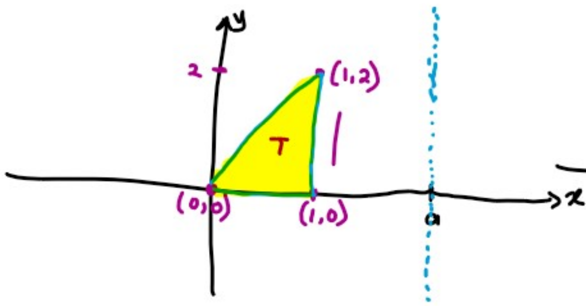
$= y-(-2)$ (So rotation is about $y=-2$)

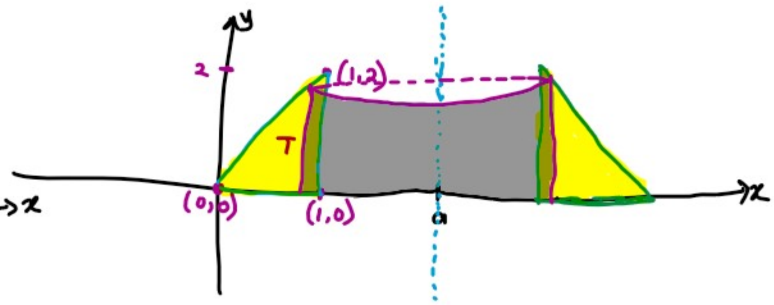
Circumference $= 2\pi(y+2)$

Height $= \frac{1}{y^2}$

Thus, by cylindrical shells method, the solid is obtained by rotating the region bounded by $y=1$, $y=4$, $x=\frac{1}{y^2}$, $x=0$ about the line $y=-2$.

44





Between (0,0) and (1,2):

$$m = \frac{y_1 - y_2}{x_1 - x_2} = \frac{0-2}{0-1} = 2$$

so $y - 0 = m(x - 0)$

$\Rightarrow y = 2x$

Radius $= a - x$

Circumference $= 2\pi(a-x)$

Height $= 2x$

$$\Longrightarrow A(x) = 2\pi(a-x)\cdot 2x$$

$$= 4\pi(ax - x^2)$$

So,

Volume of solid

$$V = \int_0^1 A(x)\,dx$$

$$= \int_0^1 4\pi(ax - x^2)\,dx$$

$$= 4\pi\left(\frac{ax^2}{2} - \frac{x^3}{3}\right)\Big|_0^1$$

$$= 4\pi\left[\left(\frac{a\cdot 1^2}{2} - \frac{1^3}{3}\right) - \left(\frac{a\cdot 0^2}{2} - \frac{0^3}{3}\right)\right]$$

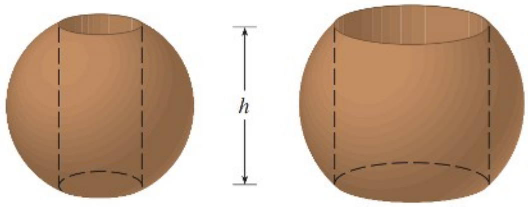
$$V = 4\pi\left(\frac{a}{2} - \frac{1}{3}\right)$$

$$\Rightarrow \frac{V}{4\pi} = \frac{a}{2} - \frac{1}{3}$$

$$\Rightarrow \frac{V}{4\pi} + \frac{1}{3} = \frac{a}{2}$$

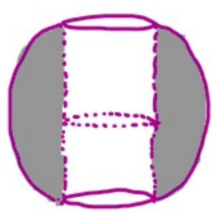
$$\Rightarrow a = \frac{V}{2\pi} + \frac{2}{3}$$

48

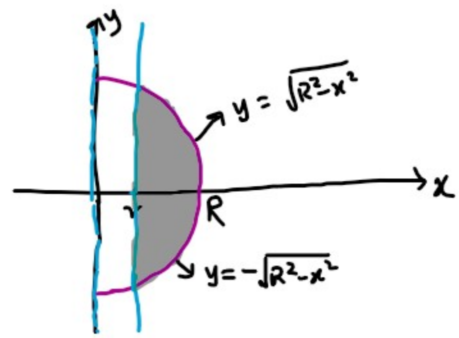


(b) Looking at the figures, my guest is that the bigger napkin has more wood in it.

(b)



Position the wooden ring so that Volume of the wood is obtained by by rotating the region bounded by $x^2+y^2=R^2$, $x=r$ about y-axis



So

$$\text{Volume of wood} = \int_r^R A(x)\,dy$$

$$= \int_r^R 4\pi x\sqrt{R^2-x^2}\,dx$$

But for any shell:

Radius $= x$

Circumference $= 2\pi x$

Height $= \sqrt{R^2-x^2} - \left(-\sqrt{R^2-x^2}\right)$

$= 2\sqrt{R^2-x^2}$

$\Rightarrow A(x) = 2\pi x\left(2\sqrt{R^2-x^2}\right)$

$= 4\pi x\sqrt{R^2-x^2}$

$u = R^2-x^2 \rightarrow \frac{du}{dx} = -2x \Rightarrow \frac{du}{-2x} = dx$

Thus,

$$\text{Volume of wood} = \int_{x=r}^{x=R} 4\pi x \cdot u^{1/2} \cdot \frac{du}{-2x}$$

$$= -2\pi\int_r^R u^{1/2}\,du$$

$$= -\frac{4\pi}{3}u^{3/2}\Big|_{x=r}^{x=R}$$

$$= -\frac{4\pi}{3}\left(R^2 - x^2\right)^{3/2}\Bigg|_r^R$$

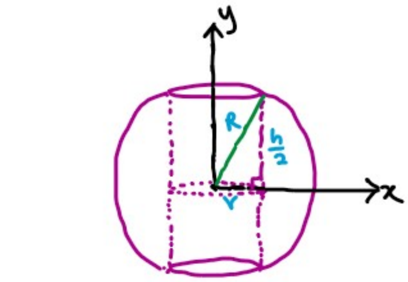
$$= -\frac{4\pi}{3}\left(R^2 - R^2\right)^{3/2} - \left(-\frac{4\pi}{3}\left(R^2 - r^2\right)^{3/2}\right)$$

$$= \frac{4\pi}{3}\left(\sqrt{R^2 - r^2}\right)^3$$

$$= \frac{4\pi}{3}\left(\sqrt{\left(\tfrac{h}{2}\right)^2}\right)^3$$

$$= \frac{4\pi}{3} \cdot \frac{h^3}{2^3}$$

$$= \frac{\pi h^3}{6}$$



By Pythagoras rule,

$$R^2 = r^2 + \left(\tfrac{h}{2}\right)^2$$

$$\Rightarrow R^2 - r^2 = \left(\tfrac{h}{2}\right)^2$$

5. A couple of weeks ago we derived the "normalization constant" for Gamma probability density functions through the following indefinite integral:

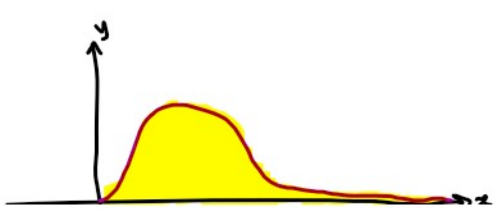
$$\int_0^\infty t^n e^{-\lambda t}\, dt = \frac{n!}{\lambda^{n+1}},$$

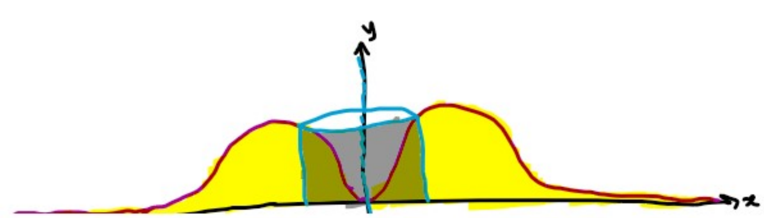
for a positive integer n (the "shape parameter") and constant $\lambda > 0$ (the "rate parameter").

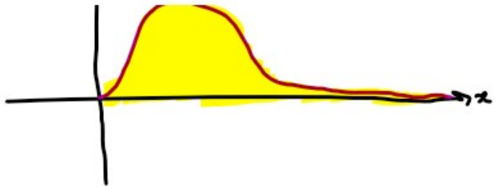
We used this result, along with the washer method, to see that the volume of the solid generated by revolving the region in the first quadrant below the curve $y = x^n e^{-\lambda x}$ about the x-axis is

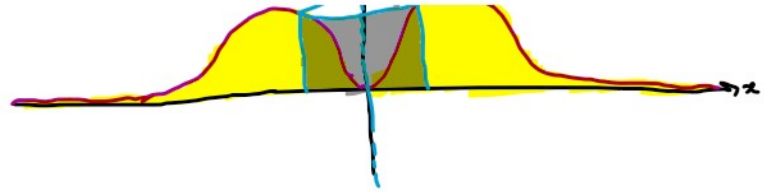
$$V = \frac{\pi(2n)!}{(2\lambda)^{2n+2}}.$$

But what is the volume of the solid generated by revolving this region about the y-axis? Can you find it using the washer method?









For any shell:

Radius $= x$

Circumference $= 2\pi x$

Height $= x^n e^{-\lambda x}$

$\Rightarrow A(x) = 2\pi x^{n+1} e^{-\lambda x}$

So,

$$\text{Volume of solid} = \int_0^\infty A(x)\,dx$$

$$= \int_0^\infty 2\pi x^{n+1} e^{-\lambda x}\,dx$$

$$= 2\pi \int_0^\infty x^{n+1} e^{-\lambda x}\,dx$$

$$= 2\pi \left(\frac{(n+1)!}{\lambda^{n+2}} \right)$$

$$= \underline{\underline{\frac{2\pi (n+1)!}{\lambda^{n+2}}}}$$

Using the washer method,

$$\text{Volume of solid} = \int_0^{\left(\frac{n}{\lambda}\right)^n e^{-n}} A(y)\,dy.$$

$x^n e^{-\lambda x}$ (except numerically!!)

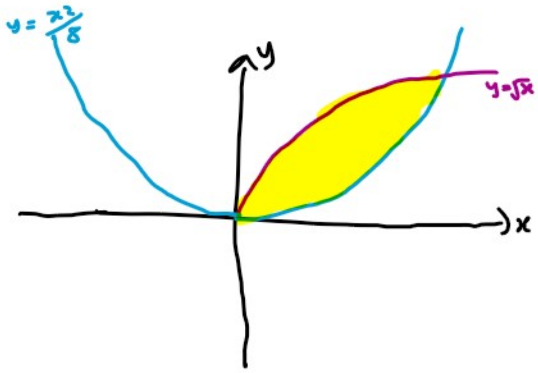
solid

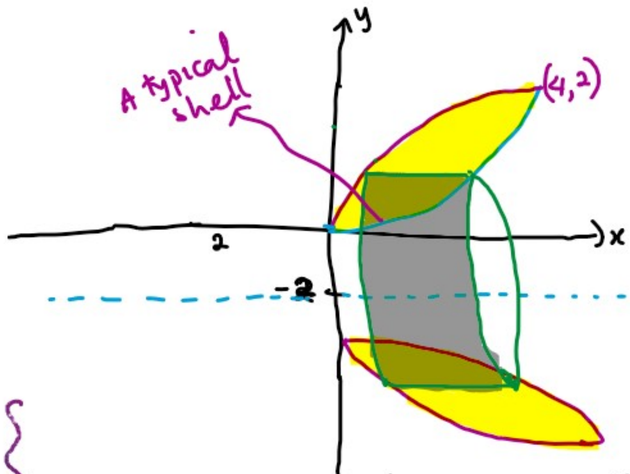
So, we have to solve for x in $y = x^n e^{-\lambda x}$ (Impossible except numerically!!)

Volumes by Cylindrical Shells

Friday, February 19, 2021

1. Consider the region in the first quadrant bounded by the curves $y = \sqrt{x}$ and $y = \frac{x^2}{8}$. What is the volume of the solid generated by revolving this region about the line $y = -2$?





Intersections:
$\sqrt{x} = y = \frac{x^2}{8}$
$\Rightarrow 8\sqrt{x} = x^2$
$\Rightarrow 64x - x^4 = 0$
$\Rightarrow x(64 - x^3) = 0$
$\Rightarrow x = 0, 4$

Since we are rotating parallel to the x-axis and using shells,

$$\text{Volume of solid} = \int_0^2 A(y)\,dy$$

For $0 < y \leq 4$, consider the shell parallel to the axis of rotation and cut at the point y. Then

Radius $= y - (-2) = y + 2$

Circumference $= 2\pi(y+2)$

Height $= \sqrt{8y} - y^2$ (NB: $y = \sqrt{x} \Rightarrow x = y^2$ and $y = \frac{x^2}{8} \Rightarrow x = \pm\sqrt{8y}$)

So

$$\begin{aligned}\text{Area of shell } A(y) &= \text{Circumference} \times \text{Height} \\ &= 2\pi(y+2)\left(\sqrt{8y} - y^2\right) \\ &= 2\pi\left(y\sqrt{8y} - y^3 + 2\sqrt{8y} - 2y^2\right) \\ &= 2\pi\left(\sqrt{8}\,y^{3/2} + 2\sqrt{8}\,y^{1/2} - y^3 - 2y^2\right)\end{aligned}$$

Thus,

$$\begin{aligned}\text{Volume of solid} &= \int_0^2 A(y)\,dy \\ &= \int_0^2 2\pi\left(\sqrt{8}\,y^{3/2} + 2\sqrt{8}\,y^{1/2} - y^3 - 2y^2\right) \\ &= 2\pi\left[\frac{2\sqrt{8}}{5}\,y^{5/2} + \frac{4\sqrt{8}}{3}\,y^{3/2} - \frac{y^4}{4} - \frac{2}{3}y^3\right]_0^2\end{aligned}$$

$$= 2\pi \left[\frac{2\sqrt{8}}{5} y^{5/2} + \frac{4\sqrt{8}}{3} y^{3/2} - \frac{y^4}{4} - \frac{2}{3} y^3\right]_0^2$$

$$= 2\pi \left[\frac{2\sqrt{8}}{5}\left(2^{5/2}\right) + \frac{4\sqrt{8}}{3}\left(2^{3/2}\right) - \frac{2^4}{4} - \frac{2}{3}\left(2^3\right)\right]$$

$$= 2\pi \left[\frac{4\sqrt{2}}{5} \cdot 2^{5/2} + \frac{8\sqrt{2}}{3} \cdot 2^{3/2} - 4 - \frac{16}{3}\right]$$

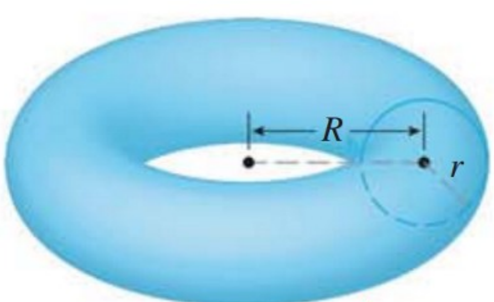
$$= 2\pi \left[\frac{4 \cdot 2^3}{5} + \frac{8 \cdot 2^2}{3} - \frac{28}{3}\right]$$

$$= 2\pi \left(\frac{32}{5} + \frac{32}{3} - \frac{28}{3}\right)$$

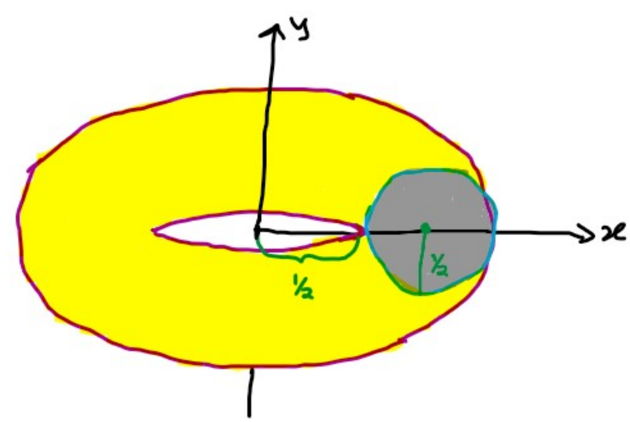
$$= 2\pi \left(\frac{32}{5} + \frac{4}{3}\right)$$

$$= 2\pi \left(\frac{96 + 20}{15}\right) = \frac{232\pi}{15}$$

2. What is the volume of a doughnut whose height is 1 inch and whose hole is 1 inch across?



A typical donut (In our case R=1 and r=1/2)

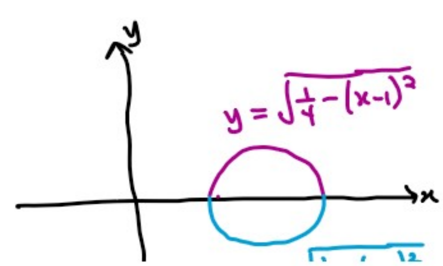


We can think of the donut as the volume swept by rotating the disk with center (1, 0) and radius ½ about the y-axis.

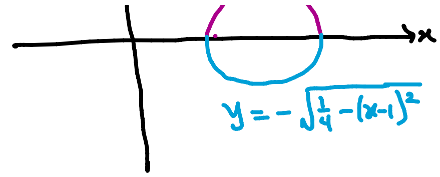
So, equation of circle that bounds this disk is

$$(x-1)^2 + y^2 = \left(\frac{1}{2}\right)^2$$

$$\Rightarrow \quad y = \pm \sqrt{\frac{1}{4} - (x-1)^2}$$



$$\Rightarrow \quad y = \pm\sqrt{\tfrac{1}{4} - (x-1)^2}$$



Thus, for any $\frac{1}{2} \le x \le \frac{3}{2}$, a shell would have:

Radius $= x$

Circumference $= 2\pi x$

Height $= \sqrt{\frac{1}{4}-(x-1)^2} - \left(-\sqrt{\frac{1}{4}-(x-1)^2}\right) = 2\sqrt{\frac{1}{4}-(x-1)^2}$

So, area of shell $A(x) = 2\pi x\left(2\sqrt{\frac{1}{4}-(x-1)^2}\right) = 4\pi x\sqrt{\frac{1}{4}-(x-1)^2}$

Hence,

$$\text{Volume of donut} = \int_{1/2}^{3/2} A(x)\,dx$$

$$= \int_{1/2}^{3/2} 4\pi x\sqrt{\tfrac{1}{4}-(x-1)^2}\,dx$$

There are many ways to evaluate this integral.

One way is to notice that

$$\int_{1/2}^{3/2} 4\pi x\sqrt{\tfrac{1}{4}-(x-1)^2}\,dx = \int_{1/2}^{3/2} 4\pi(x-1+1)\sqrt{\tfrac{1}{4}-(x-1)^2}\,dx$$

$$= \underbrace{\int_{1/2}^{3/2} 4\pi(x-1)\sqrt{\tfrac{1}{4}-(x-1)^2}\,dx}_{\mathrm{I}} + \underbrace{\int_{1/2}^{3/2} 4\pi\sqrt{\tfrac{1}{4}-(x-1)^2}\,dx}_{\mathrm{II}}$$

Then, using $u = \frac{1}{4}-(x-1)^2$ for I $\Rightarrow \frac{du}{dx} = -2(x-1) \Rightarrow \frac{du}{-2(x-1)} = dx$

Also, $x = \frac{1}{2} \Rightarrow u = \frac{1}{4} - \left(\frac{1}{2}-1\right)^2 = \frac{1}{4} - \frac{1}{4} = 0$

$x = \frac{3}{2} \Rightarrow u = \frac{1}{4} - \left(\frac{3}{2}-1\right)^2 = \frac{1}{4} - \frac{1}{4} = 0$

This alone guarantees that $\int_{1/2}^{3/2} 4\pi(x-1)\sqrt{\frac{1}{4}-(x-1)^2}\,dx = 0.$

So,

$$\int_{1/2}^{3/2} 4\pi(x-1)\sqrt{\frac{1}{4}-(x-1)^2}\,dx = 0.$$

For II, we notice that

$$\int_{1/2}^{3/2} 4\pi\sqrt{\frac{1}{4}-(x-1)^2}\,dx = 4\pi\int_{1/2}^{3/2}\sqrt{\frac{1}{4}-(x-1)^2}\,dx$$

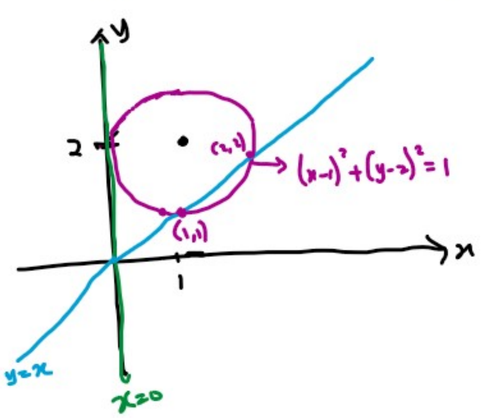
$$= 4\pi\,(\text{Area of semi-circle bounding the disk which has radius } 1/2)$$

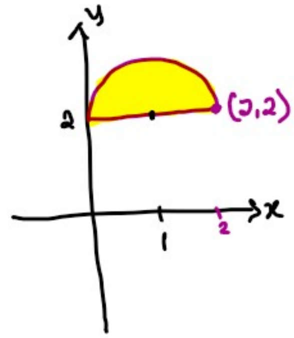
$$= 4\pi\left(\frac{\pi}{2}\left(\frac{1}{2}\right)^2\right)$$

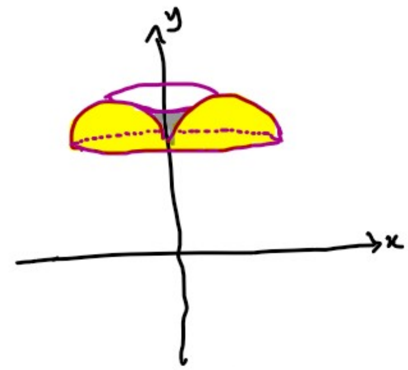
$$= \frac{\pi^2}{2}$$

$\therefore$ Volume of the donut $= \text{I} + \text{II} = 0 + \frac{\pi^2}{2} = \frac{\pi^2}{2}$

3. Consider the region bounded by the y-axis, the line $y = x$, and the upper half of the circle given by the equation $(x-1)^2 + (y-2)^2 = 1$. Describe the shape of the solid generated by rotating this region about the y-axis, and find its volume.







Solid looks like two equal mountains that meet at their ground levels.

mountains that meet at their ground levels.

For any $0 < x \le 2$, a cylindrical shell would have:

Radius $= x$

Circumference $= 2\pi x$

Height $= \left(2 + \sqrt{1-(x-1)^2}\right) - 2$

$= \sqrt{1-(x-1)^2}$

NB: $(x-1)^2 + (y-2)^2 = 1 \Rightarrow y = 2 \pm \sqrt{1-(x-1)^2}$

So, area of a shell

$$A(x) = 2\pi x \sqrt{1-(x-1)^2}$$

Thus,

$$\text{Volume of solid} = \int_0^2 A(x)\,dx$$

$$= \int_0^2 2\pi x \sqrt{1-(x-1)^2}\,dx$$

$$= 2\pi \int_0^2 (x-1+1)\sqrt{1-(x-1)^2}\,dx$$

same idea of integration as was in the previous problem.

$$= \underbrace{2\pi \int_0^2 (x-1)\sqrt{1-(x-1)^2}\,dx}_{\text{I}} + \underbrace{2\pi \int_0^2 \sqrt{1-(x-1)^2}\,dx}_{\text{II}}$$

I: Again this is zero by u-sub

Furthermore, we could use trigonometric substitution to evaluate II, but an easier way is to notice that $\int_0^2 \sqrt{1-(x-1)^2}\,dx$ is the area under the upper semi-circle $y = \sqrt{1-(x-1)^2}$ (which has radius 1)

So

$$\text{II} = 2\pi \int_0^2 \sqrt{1-(x-1)^2}\,dx$$

$= 2\pi \left(\frac{\pi}{2}(1)^2 \right)$

$= \pi^2$

Hence,

Volume of solid $= I + II = 0 + \pi^2 = \underline{\underline{\pi^2}}$