Work-6.4: #4, 16, 17, 21, 23-24, 32-34

Tuesday, February 23, 2021

(4) force $f(x) = \cos\left(\frac{\pi x}{3}\right)$, $x = 1$ to $x = 2$.

$$\text{Work done} = \int_1^2 f(x)\,dx$$

$$= \int_1^2 \cos\left(\frac{\pi x}{3}\right) dx$$

$$= \frac{3}{\pi} \sin\left(\frac{\pi x}{3}\right)\Big|_1^2$$

$$= \frac{3}{\pi}\left[\sin\left(\frac{2\pi}{3}\right) - \sin\left(\frac{\pi}{3}\right)\right]$$

$$= \frac{3}{\pi}\left[\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2}\right]$$

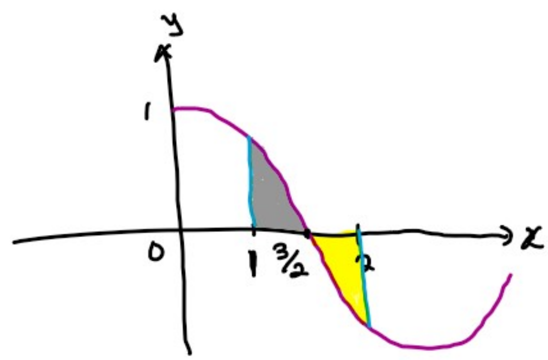
$$= \frac{3}{\pi}(0)$$

$$= 0 \text{ Joules}$$

For $1 \le x \le 2$:

$\cos\left(\frac{\pi x}{3}\right) = 0 \Rightarrow \frac{\pi x}{3} = \frac{\pi}{2} \Rightarrow x = 3/2$

Since $\int_1^2 \cos\left(\frac{\pi x}{3}\right) dx = \int_1^{3/2} \cos\left(\frac{\pi x}{3}\right) dx + \int_{3/2}^2 \cos\left(\frac{\pi x}{3}\right) dx$



and areas of the regions in gray and in Yellow are equal and opposite, we conclude that the particle accelerates between $x=1$ to $x=1.5$ by doing the work $\int_1^{3/2} \cos\left(\frac{\pi x}{3}\right) dx$ but decelerates between $x = 1.5$ to $x = 2$ by doing the work $\int_{3/2}^2 \cos\left(\frac{\pi x}{3}\right) dx = -\int_1^{3/2} \cos\left(\frac{\pi x}{3}\right) dx$.

(16) To find the work required to raise one end of the chain, we either assume the chain is fixed at one end or the force required to move unraised part of the chain on the ground is negligible.

Here, we assume one end is fixed:

The force required to lift the chain x meters above the ground is equal and opposite to the weight of the chain at x meters high.

So,

$$\text{force } f(x) = \text{mass} \times \text{acceleration}$$

$$= 8x \times 9.8$$

$$= 78.4x$$

But mass of x meters of the chain $= x\left(\frac{80}{10}\,\text{kg/m}\right) = 8x\,\text{kg}$, assuming the 80kg mass is uniformly spread over the chain.

Thus,

$$\text{work required} = \int_0^6 f(x)\,dx$$

$$= \int_0^6 78.4x\,dx$$

$$= \left.\frac{78.4x^2}{2}\right|_0^6$$

$$= \frac{78.4}{2}\left(6^2 - 0^2\right)$$

$$= 78.4(18)$$

$$= 1411.2\,J.$$

(17) The force pulling the bucket upward is equal and opposite to the weight of the system.

At x m above the ground, $(0.8\,kg/m)x = 0.8x$ kg of rope is used to pull $\left(\frac{36}{12}kg/m\right)x = 3x$ kg of water. Since the bucket has a mass of 10kg, then at x meters high, the force has pulled

$$0.8x + 3x + 10 = (3.8x + 10)kg \quad \text{upward.}$$

So at x meters high, force applied is

$$f(x) = mg = (3.8x+10)9.8 = (37.24x + 98)$$

$\Rightarrow$ Work done pulling the bucket to the top $= \int_0^{12} (37.24x + 98)\,dx$

$$= \frac{37.24x^2}{2} + 98x \Big|_0^{12}$$

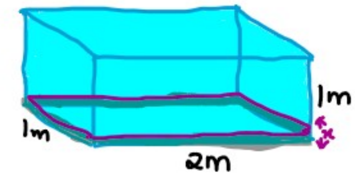
$$= \frac{37.24(12^2)}{2} + 98(12) - \left(\frac{37.24(0^2)}{2} + 98(0)\right)$$

$$= \frac{5362.56}{2} + 1176$$

$$= 2681.28 + 1176$$

$$= 3857.28\,J$$

(21) Let x m $(0 < x \le 1/2)$ be an infinitesimal volume of water pumped out at Δx thickness. Then, volume of water pumped out at x m is



$$2 \times x \times \Delta x = 2x \Delta x \text{ m}^3$$

$\Rightarrow$ mass of this water $= 2x\Delta x (1000 \text{ kg/m}^3)$ $\quad 2000x\Delta x \text{ Kg}$

So

$$f(x) = 2000x\Delta x \times 9.8 = 19600x\Delta x$$

is the force that pumped out water at xm height

Thus,

Work needed to pump out half of the water $= \int_0^{1/2} 19600x\,dx$

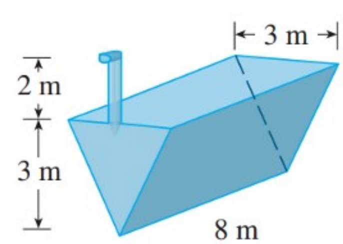
$$= \left.\frac{19600x^2}{2}\right|_0^{1/2}$$

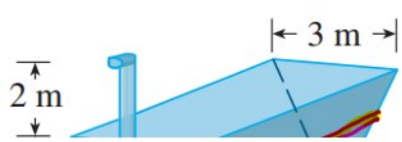
$$= 9800\left(\left(\tfrac{1}{2}\right)^2 - 0^2\right)$$

$$= \frac{9800}{4}$$

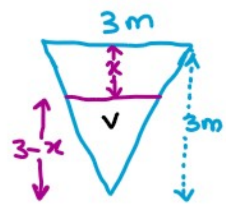
$$= 2450 \text{J}$$

(23) Consider a horizontal infinitesimal slice of length Δx at depth xm.





Let's call the width of this slice v. Then using similar triangles:



Comparing heights and widths, we have:

$$\frac{3-x}{v} = \frac{3}{3}$$

$$\Rightarrow \quad v = 3-x$$

Volume of this slice $= 8(3-x)(\Delta x)$.

Mass of this slice $=$ density $\times$ volume

$= 1000 \times 8(3-x)\Delta x$

$= 8000(3-x)\Delta x$

Force required on slice $f(x) = mg = 8000(3-x)\Delta x \times 9.8$

But distance of this slice from the spout is $x+2$. So

work done on slice $= 78400(3-x)(x+2)\Delta x$.

Hence,

Work required to pump all water out of spout $= \int_0^3 78400(3-x)(x+2)\,dx$

$$= 78400 \int_0^3 (3x+6-x^2-2x)\,dx$$

$$= 78400 \int_0^3 (6 + x - x^2)\, dx$$

$$= 78400 \left(6x + \frac{x^2}{2} - \frac{x^3}{3}\right)\Bigg|_0^3$$

$$= 78400 \left(6(3) + \frac{3^2}{2} - \frac{3^3}{3}\right)$$

$$= 78400 \left(18 + \frac{9}{2} - 9\right)$$

$$= 78400 \left(\frac{27}{2}\right)$$

$$= 1058400\, \text{J}$$

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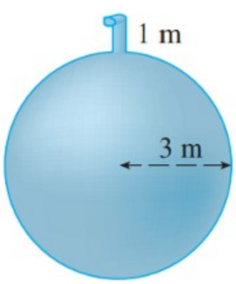
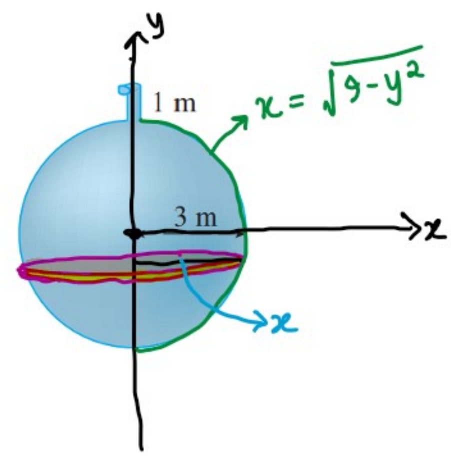


Align a coordinate system to pass through the center of the tank as shown and

Consider an infinitesimal slice with

Radius $= x$

Height $= \Delta y$



So volume of slice

$$V = \pi x^2 \Delta y$$

$$= \pi \left(\sqrt{9 - y^2}\right)^2 \Delta y$$

$$= \pi (9 - y^2)\, \Delta y$$

$\Rightarrow$ mass of the slice $=$ density $\times$ volume

$\Rightarrow$ mass of the slice = density x volume

$$= 1000 \times \pi(9-y^2)\Delta y$$

$$= 1000\pi(9-y^2)\Delta y$$

Force required to pump water in slice $f(x) = mg$

$$= 1000\pi(9-y^2)\Delta y \times 9.8$$

$$= 9800\pi(9-y^2)\Delta y$$

Work done on slice = force x distance

$$= 9800\pi(9-y^2)\Delta y \times (4-y)$$

$$= 9800\pi(9-y^2)(4-y)\Delta y$$

Hence,

Work required to pump all water $= \int_{-3}^{3} 9800\pi(9-y^2)(4-y)\,dy$

$$= 9800\pi \int_{-3}^{3} (36-9y-4y^2+y^3)\,dy$$

$$= 9800\pi \left(36y - \frac{9}{2}y^2 - \frac{4}{3}y^3 + \frac{1}{4}y^4\right)\Big|_{-3}^{3}$$

$$= 9800\pi\left(\left(36(3) - \frac{9}{2}(3^2) - \frac{4}{3}(3^3) + \frac{1}{4}(3^4)\right) - \left(36(-3) - \frac{9}{2}(-3)^2 - \frac{4}{3}(-3)^3 + \frac{1}{4}(-3)^4\right)\right)$$

$$= 9800\pi\left(\left(108 - \frac{81}{2} - \frac{108}{3} + \frac{81}{4}\right) - \left(-108 - \frac{81}{2} + \frac{108}{3} + \frac{81}{4}\right)\right)$$

$$= 9800\pi\left(108 - \frac{81}{2} - \frac{108}{3} + \frac{81}{4} + 108 + \frac{81}{2} - \frac{108}{3} - \frac{81}{4}\right)$$

$$= 9800\pi\left(216 - \frac{216}{3}\right)$$

$$= 9800\pi\left(216 - 72\right)$$

$$- \quad 9800\pi\,(144)$$

$$41 \quad 00\pi$$

(32) Force $f(x) = 5.7x^2 + 1.5x$

Work done on the car $= \int_0^{60} (5.7x^2 + 1.5x)\,dx$

$$= \frac{5.7}{3}x^3 + \frac{1.5}{2}x^2 \Big|_0^{60}$$

$$= \frac{5.7}{3}(60^3) + \frac{1.5}{2}(60^2)$$

$$= 413100.$$

But

Work $= \frac{1}{2}mv^2$ from Exercise 31 (b)

So

$$\frac{1}{2}(800)v^2 = 413100$$

So

$$\frac{1}{2}(800)v^2 = 413100$$

$$\Rightarrow \quad v = \sqrt{\frac{2(413100)}{800}}$$

$$= 32.14 \text{ m/s}.$$

33 $F = \dfrac{G m_1 m_2}{r^2}$, $r = a$ to $r = b$

b Work needed $= \displaystyle\int_a^b \frac{G m_1 m_2}{r^2}\, dr$

$$= G m_1 m_2 \int_a^b \frac{1}{r^2}\, dr$$

$$= G m_1 m_2 \left(-\frac{1}{r}\right)\Big|_a^b$$

$$= G m_1 m_2 \left(-\frac{1}{b} + \frac{1}{a}\right)$$

$$= G m_1 m_2 \left(\frac{1}{a} - \frac{1}{b}\right) \text{ J}.$$

b $m_1 = 1000\text{kg}$, $m_2 = 5.98 \times 10^{24}\text{kg}$, $r = 6.37 \times 10^6\text{m}$ to
$r = 6.37 \times 10^6\text{m} + 1000 \times 10^3\text{m}$ ($1\text{Km} = 1000\text{m}$), $G = 6.67 \times 10^{-11}\ \text{Nm}^2/\text{kg}^2$.

So by (a) above,

$$\text{Work} = Gm_1m_2\left(\frac{1}{a} - \frac{1}{b}\right)$$

$$= \left(6.67\times10^{-11}\right)\left(1000\right)\left(5.98\times10^{24}\right)\left(\frac{1}{6.37\times10^{6}} - \frac{1}{7.37\times10^{6}}\right)$$

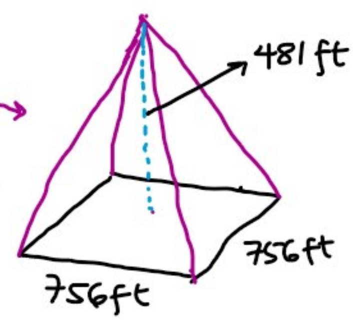
$$= 8.4961694343\times10^{9}$$

$$\approx 8.50\times10^{9}$$

(34)

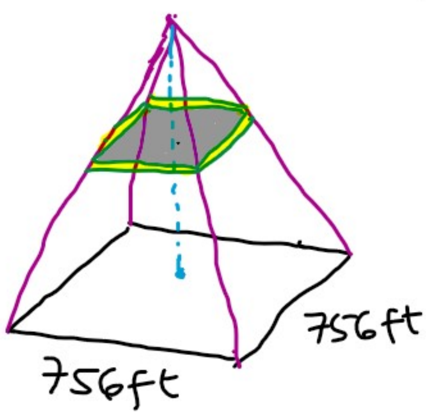
(a)

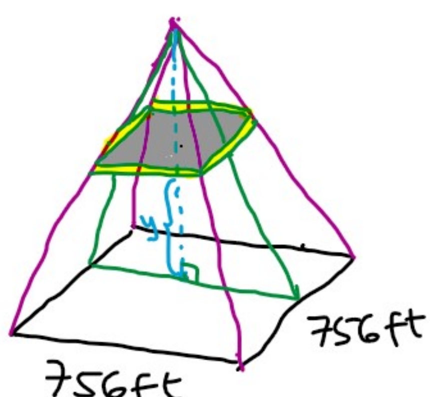
Assume the pyramid has smooth sides as shown

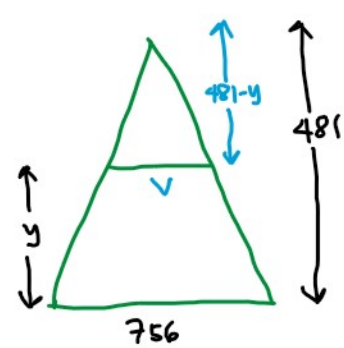


Consider a horizontal slab of thickness Δy at height y ft from the base square.

The slab will have a square shape since the base of the pyramid is a square







By similar Triangles

$$\frac{v}{481-y} = \frac{756}{481}$$

$$\frac{v}{481-y} = \frac{756}{481}$$

$$\Rightarrow v = \frac{756}{481}(481-y)$$

So,

Volume of slab $= \left(\frac{756}{481}(481-y)\right)^2 \Delta y$ since the slab has a square shape.

$\Rightarrow$ mass of slab = density x volume

$$= 150\left(\frac{756}{481}(481-y)\right)^2 \Delta y \text{ lb}$$

$\Rightarrow$ Force acting on slab $= 150\left(\frac{756}{481}(481-y)\right)^2 \Delta y$ (Notice we do not multiply by gravity since pound is a unit of force)

$\Rightarrow$ Work done on slab = force x distance

$$= 150\left(\frac{756}{481}(481-y)\right)^2 \Delta y \times y$$

since the slab is y ft above the ground (base)

$$= 150y\left(\frac{756}{481}(481-y)\right)^2 \Delta y.$$

Hence,

Total work done in building the pyramid $= \int_0^{481} 150y\left(\frac{756}{481}(481-y)\right)^2 dy$

$$= 150\left(\frac{756}{481}\right)^2 \int_0^{481} y\left(231361 - 962y + y^2\right) dy$$

$$= 150\left(\frac{756}{481}\right)^2 \int_0^{481} \left(231361y - 962y^2 + y^3\right) dy$$

$$= 150\left(\frac{756}{481}\right)^2 \left(\frac{231361}{2}y^2 - \frac{962}{3}y^3 + \frac{1}{4}y^4\right)\Bigg|_0^{481}$$

$$= 150\left(\frac{756}{481}\right)^2 \left(\frac{231361}{2}(481^2) - \frac{962}{3}(481^3) + \frac{1}{4}(481^4)\right)$$

$$\approx 1.6529 \times 10^{12} \text{ ft-lb}$$

(b) In 20 yrs, each laborer worked

$$= 20\text{yrs} \times 340\text{days} \times 10 \text{ hrs} = 68000 \text{ hrs.}$$

So total work for each laborer in 20 yrs

$$= 200 \text{ ft-lb/h} \times 68000\text{h}$$

$$= 13600000 \text{ ft-lb}$$

Hence

$$\#\text{ of laborers required} = \frac{\text{Total work}}{\text{work for each laborer}} = \frac{1.6529 \times 10^{12}}{13600000}$$

$$\approx 121536 \text{ laborers.}$$

2. A spring scale is compressed 1 cm when a 50 kg rock is placed on it.

   (a) What is the weight of a rock that compresses the scale 2 cm? (Assume that Hooke's law applies to the spring.)

   (b) How much work is done compressing the scale 2 cm?

$F_1 = mg = 50(9.8) = 490$

$x_1 = 1\text{cm}.$

(a) $x_2 = 2\text{cm}, \ F_2 = ?$

By Hooke's law:

$$\frac{F_2}{x_2} = \frac{F_1}{x_1}$$

$$\Rightarrow \frac{F_2}{2} = \frac{490}{1}$$

$$\Rightarrow F_2 = 2(490) = 980\text{N}$$

Hence, the weight of the rock that compresses the scale 2cm is 980N (or the rock has mass of 100 kg $= \frac{980}{9.8}$)

(b) Work = force × distance

$= 980 \times 0.02$

$= 19.6 \text{ J}$