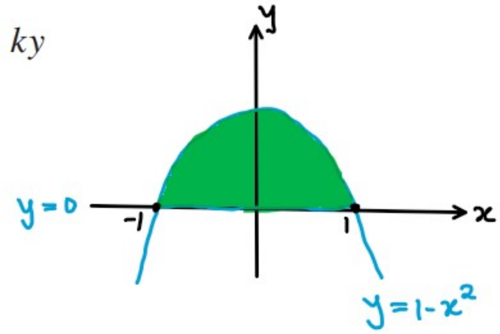
Section 15.4: 7, 29

3–10 Find the mass and center of mass of the lamina that occupies the region D and has the given density function ρ.

7. D is bounded by $y = 1 - x^2$ and $y = 0$; $\rho(x, y) = ky$



A type I region description for D is

$D = \{(x,y): -1 \le x \le 1,\ 0 \le y \le 1 - x^2\}$

Thus, the mass

$$
\begin{aligned}
m &= \iint_D \rho(x,y)\,dA \\
&= \int_{-1}^{1}\int_{0}^{1-x^2} ky\,dy\,dx \\
&= k\int_{-1}^{1}\left[\frac{y^2}{2}\right]_{y=0}^{y=1-x^2} dx \\
&= \frac{k}{2}\int_{-1}^{1}(1-x^2)^2\,dx \\
&= \frac{k}{2}\int_{-1}^{1}(1-2x^2+x^4)\,dx \\
&= \frac{k}{2}\left[x - \frac{2}{3}x^3 + \frac{1}{5}x^5\right]_{x=-1}^{x=1} \\
&= \frac{k}{2}\left[\left(1 - \frac{2}{3}(1^3) + \frac{1}{5}(1^5)\right) - \left((-1) - \frac{2}{3}(-1)^3 + \frac{1}{5}(-1)^5\right)\right] \\
&= \frac{k}{2}\left[\left(1 - \frac{2}{3} + \frac{1}{5}\right) - \left(-1 + \frac{2}{3} - \frac{1}{5}\right)\right] \\
&= \frac{k}{2}\left[\left(\frac{15-10+3}{15}\right) - \left(\frac{-15+10-3}{15}\right)\right] \\
&= \frac{k}{2}\left(\frac{8}{15} + \frac{8}{15}\right) \\
&= \frac{8k}{15}.
\end{aligned}
$$

The coordinate of the center of mass of the lamina is $(\bar{x}, \bar{y})$ where

$$\bar{x} = \frac{M_y}{m} \quad \text{and} \quad \bar{y} = \frac{M_x}{m}$$

and

$$M_y = \iint_D x\rho(x,y)\,dA$$

$$= \int_{-1}^{1}\int_{0}^{1-x^2} kxy\,dy\,dx$$

$$= k\int_{-1}^{1}\left[\frac{1}{2}xy^2\right]_{y=0}^{y=1-x^2} dx$$

$$= \frac{k}{2}\int_{-1}^{1} x(1-x^2)^2\,dx$$

$$u = 1-x^2 \Rightarrow du = -2x\,dx$$
$$\Rightarrow dx = \frac{du}{-2x}$$

$$= \frac{k}{2}\int_{-1}^{1} x\cdot u^2\cdot \frac{du}{-2x}$$

$$= -\frac{k}{4}\int_{-1}^{1} u^2\,du$$

$$= -\frac{k}{4}\left[\frac{1}{3}u^3\right]_{x=-1}^{x=1}$$

$$= \frac{-k}{12}\left[(1-x^2)^3\right]_{x=-1}^{x=1}$$

$$= -\frac{k}{12}\left[(1-(1)^2)^3 - (1-(-1)^2)^3\right]$$

$$= -\frac{k}{12}\left[0^3 - 0^3\right]$$

$$= 0$$

$$M_x = \iint_D y\rho(x,y)\,dA$$

$$= \int_{-1}^{1}\int_{0}^{1-x^2} ky^2\,dy\,dx$$

$$= k\int_{-1}^{1}\left[\frac{1}{3}y^3\right]^{y=1-x^2} dx$$

$$= k\int_{-1}^{1}\left[\frac{1}{3}y^3\right]_{y=0}^{y=1-x^2}dx$$

$$= \frac{k}{3}\int_{-1}^{1}(1-x^2)^3\,dx$$

$$(1-x^2)^3 = (1-x^2)(1-x^2)^2$$
$$= (1-x^2)(1-2x^2+x^4)$$
$$= 1-2x^2+x^4-x^2+2x^4-x^6$$
$$= 1-3x^2+3x^4-x^6$$

$$= \frac{k}{3}\int_{-1}^{1}(1-3x^2+3x^4-x^6)\,dx$$

$$= \frac{k}{3}\left[x-x^3+\frac{3}{5}x^5-\frac{1}{7}x^7\right]_{-1}^{1}$$

$$= \frac{k}{3}\left[\left(1-1+\frac{3}{5}-\frac{1}{7}\right)-\left(-1+1-\frac{3}{5}+\frac{1}{7}\right)\right]$$

$$= \frac{k}{3}\left(\frac{3}{5}+\frac{3}{5}-\frac{1}{7}-\frac{1}{7}\right)$$

$$= \frac{k}{3}\left(\frac{6}{5}-\frac{2}{7}\right)$$

$$= \frac{k}{3}\left(\frac{42-10}{35}\right)$$

$$= \frac{32k}{105}.$$

So

$$\bar{x} = \frac{M_y}{m} = \frac{0}{\frac{8k}{15}} = 0$$

$$\bar{y} = \frac{M_x}{m} = \frac{32k}{105} \div \frac{8k}{15} = \frac{32k}{105}\times\frac{15}{8k} = \frac{4}{7}$$

Hence, the mass $m = \frac{8k}{15}$ and the center of mass $(\bar{x},\bar{y}) = \left(0, \frac{4}{7}\right)$.

29. Suppose X and Y are random variables with joint density function

$$f(x, y) = \begin{cases} 0.1e^{-(0.5x+0.2y)} & \text{if } x \geqslant 0,\ y \geqslant 0 \\ 0 & \text{otherwise} \end{cases}$$

(a) Verify that f is indeed a joint density function.

Since $f(x,y) \geq 0$, it suffices to show $\iint_{\mathbb{R}^2} f(x,y)\,dA = 1.$

But

$$\iint_{\mathbb{R}^2} f(x,y)\,dA = \int_0^\infty \int_0^\infty 0.1e^{-(0.5x+0.2y)}\,dx\,dy$$

$$= \int_0^\infty \lim_{t\to\infty} \int_0^t 0.1\,e^{-(0.5x+0.2y)}\,dx\,dy$$

$$= \int_0^\infty \lim_{t\to\infty} \left[\frac{0.1}{-0.5}\,e^{-(0.5x+0.2y)}\right]_{x=0}^{x=t} dy$$

$$= -\frac{1}{5}\int_0^\infty \lim_{t\to\infty} \left[e^{-(0.5t+0.2y)} - e^{-0.2y}\right] dy$$

$$= -\frac{1}{5}\int_0^\infty \lim_{t\to\infty} e^{-0.2y}\left[e^{-0.5t} - 1\right] dy$$

$$= -\frac{1}{5}\int_0^\infty e^{-0.2y} \lim_{t\to\infty} \left[e^{-0.5t} - 1\right] dy$$

$$= -\frac{1}{5}\int_0^\infty e^{-0.2y}\,(0-1)\,dy$$

$$= \frac{1}{5}\int_0^\infty e^{-0.2y}\,dy$$

$$= \frac{1}{5}\left[\lim_{t\to\infty}\int_0^t e^{-0.2y}\,dy\right]$$

$$= \frac{1}{5}\lim \left[\frac{-1}{\ldots}\,e^{-0.2y}\right]^{y=t}$$

$$= \frac{1}{5} \lim_{t \to \infty} \left[\frac{-1}{0.2} e^{-0.2y} \right]_{y=0}^{y=t}$$

$$= -1 \left(\lim_{t \to \infty} e^{-0.2t} - e^{0} \right)$$

$$= -1 \, (0 - 1)$$

$$= 1.$$

Hence, f is indeed a joint density.

(b) Find the following probabilities.

(i) $P(Y \geqslant 1)$ (ii) $P(X \leqslant 2, Y \leqslant 4)$

(i) $$P(Y \geqslant 1) = \int_{1}^{\infty} \int_{-\infty}^{\infty} f(x,y) \, dx\, dy$$

$$= \int_{1}^{\infty} \int_{0}^{\infty} 0.1 e^{-(0.5x + 0.2y)} \, dx\, dy$$

$$= \int_{1}^{\infty} \left[\frac{0.1}{-0.5} e^{-(0.5x+0.2y)} \right]_{x=0}^{\infty} dy$$

$$= -\frac{1}{5} \int_{1}^{\infty} e^{-0.2y} \left[e^{-0.5x} \right]_{x=0}^{\infty} dy$$

$$= -\frac{1}{5} \int_{1}^{\infty} e^{-0.2y} \left[0 - 1 \right] dy$$

$$= \frac{1}{5} \int_{1}^{\infty} e^{-0.2y} \, dy$$

$$= \left[e^{-0.2y} \right]^{\infty}$$

$$= \frac{1}{5}\left[\frac{1}{-0.2} e^{-0.2y}\right]_{y=1}^{\infty}$$

$$= -1\left[e^{-0.2y}\right]_{y=1}^{\infty}$$

$$= -1\left(0 - e^{-0.2}\right)$$

$$= e^{-0.2}$$

$$\approx 0.8187$$

(ii) $P(X \leq 2, Y \leq 4) = \int_{-\infty}^{2}\int_{-\infty}^{4} f(x,y)\,dy\,dx$

$$= \int_{0}^{2}\int_{0}^{4} 0.1 e^{-(0.5x+0.2y)}\,dy\,dx$$

$$= \int_{0}^{2}\left[\frac{0.1}{-0.2} e^{-(0.5x+0.2y)}\right]_{y=0}^{4} dx$$

$$= -\frac{1}{2}\int_{0}^{2} e^{-0.5x}\left[e^{-0.2(4)} - e^{0}\right] dx$$

$$= -\frac{1}{2}\int_{0}^{2} e^{-0.5x}\left(e^{-0.8} - 1\right) dx$$

$$= -\frac{1}{2}\left(e^{-0.8} - 1\right)\int_{0}^{2} e^{-0.5x}\,dx$$

$$= \frac{1}{2}\left(1 - e^{-0.8}\right)\left[\frac{1}{-0.5} e^{-0.5x}\right]_{x=0}^{x=2}$$

$$= -1\left(1 - e^{-0.8}\right)\left[e^{-0.5(2)} - e^{0}\right]$$

$$= -\left(1-e^{-0.8}\right)\left(e^{-1}-1\right)$$

$$= \left(1-e^{-0.8}\right)\left(1-e^{-1}\right)$$

$$\approx 0.3481.$$

(c) Find the expected values of X and Y.

The expected value of X

$$EX = \iint_{\mathbb{R}^2} x f(x,y)\, dA$$

$$= \int_0^\infty \int_0^\infty 0.1 x e^{-(0.5x+0.2y)}\, dy\, dx$$

$$= 0.1 \int_0^\infty x e^{-0.5x} \left[\frac{1}{-0.2} e^{-0.2y} \right]_{y=0}^{\infty} dx$$

$$= -\frac{1}{2} \int_0^\infty x e^{-0.5x} \left[0 - 1 \right] dx$$

$$= \frac{1}{2} \int_0^\infty x e^{-0.5x}\, dx$$

$$= \frac{1}{2} \left[-2x e^{-0.5x} \Big|_0^\infty - 4 e^{-0.5x} \Big|_0^\infty \right]$$

$$= \frac{1}{2} \left[(0+0) - (0-4) \right]$$

$$= \frac{4}{2} = 2.$$

	u	dv
$+$	x	$e^{-0.5x}$
$-$	1	$-2e^{-0.5x}$
$+$	0	$4e^{-0.5x}$

Integration by parts

2

NB: $x e^{-0.5x}\Big|^{\infty} = 0$ since $\lim\limits_{t\to\infty} \dfrac{t}{e^{0.5t}} \overset{L'H}{=} \lim\limits_{t\to\infty} \dfrac{2}{e^{0.5t}} = 0.$

Similarly, the expected value of Y

$$\mathbb{E}Y = \iint_{\mathbb{R}^2} y\, f(x,y)\, dA$$

$$= \int_0^{\infty}\int_0^{\infty} 0.1\, y\, e^{-(0.5x+0.2y)}\, dx\, dy$$

$$= 0.1 \int_0^{\infty} y e^{-0.2y}\, dy \int_0^{\infty} e^{-0.5x}\, dx$$

$$= 0.1 \underbrace{\left[-5ye^{-0.2y}\Big|_0^{\infty} - 25e^{-0.2y}\Big|_0^{\infty}\right]}_{\text{From integration by parts}} \left[-2e^{-0.5x}\Big|_0^{\infty}\right]$$

	u	dv
+	y	$e^{-0.2y}$
−	1	$-5e^{-0.2y}$
+	0	$25e^{-0.2y}$

Integration by parts

$$= 0.1\left[(0+0) - (0-25)\right]\left[0-(-2)\right]$$

$$= 0.1\,(25)\,(2)$$

$$= 5.$$

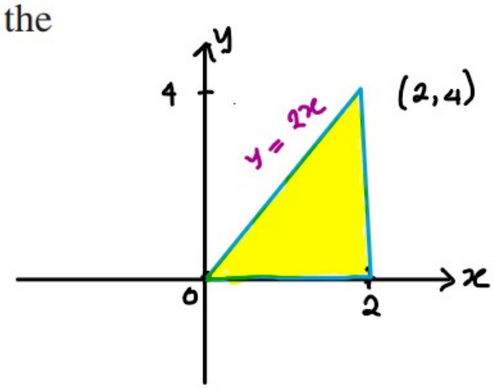
NB: $y e^{-0.2y}\Big|^{\infty} = 0$ since $\lim\limits_{t\to\infty} \dfrac{t}{e^{0.2t}} \overset{L'H}{=} \lim\limits_{t\to\infty} \dfrac{5}{e^{0.2t}} = 0.$

$m = \frac{4-0}{2-0} = 2$

$\Rightarrow y = 2x$

1–12 Find the area of the surface.

4. The part of the surface $2y + 4z - x^2 = 5$ that lies above the triangle with vertices (0, 0), (2, 0), and (2, 4)



$D = \{(x,y) : 0 \le x \le 2,\ 0 \le y \le 2x\}$

$2y + 4z - x^2 = 5$

$\Rightarrow z = \frac{1}{4}(5 + x^2 - 2y)$

$\Rightarrow z_x = \frac{1}{4}(2x) = \frac{x}{2} \Rightarrow z_x^2 = \frac{x^2}{4}$

$z_y = \frac{1}{4}(-2) = -\frac{1}{2} \Rightarrow z_y^2 = \frac{1}{4}$

$$\Rightarrow \sqrt{1 + z_x^2 + z_y^2} = \sqrt{1 + \frac{x^2}{4} + \frac{1}{4}}$$

$$= \sqrt{\frac{5 + x^2}{4}}$$

$$= \frac{1}{2}\sqrt{5 + x^2}$$

Thus, surface area

$$A(s) = \iint_D \sqrt{1 + f_x^2 + f_y^2}\, dA$$

$$= \frac{1}{2}\int_0^2 \int_0^{2x} \sqrt{5 + x^2}\, dy\, dx$$

$$= \frac{1}{2}\int_0^2 \sqrt{5 + x^2} \int_0^{2x} dy\, dx$$

$$= \frac{1}{2}\int_0^2 \sqrt{5 + x^2}\, \Big[y\Big]_{y=0}^{y=2x} dx$$

$$= \frac{1}{2} \int_0^2 2x\sqrt{5+x^2}\, dx$$

$$= \frac{1}{2} \int_5^9 2x \cdot u^{1/2} \cdot \frac{du}{2x}$$

$$= \frac{1}{2} \int_5^9 u^{1/2} du$$

$$= \frac{1}{2} \cdot \frac{2}{3} u^{3/2} \Big|_5^9$$

$$= \frac{1}{3} \left(9^{3/2} - 5^{3/2}\right)$$

$$= \frac{1}{3} \left(27 - 5\sqrt{5}\right).$$

$u = 5 + x^2 \Rightarrow du = 2x dx$

$\Rightarrow dx = \frac{du}{2x}$

$x = 0 \Rightarrow u = 5$

$x = 2 \Rightarrow u = 5 + 2^2 = 9$

3–8 Evaluate the iterated integral.

5. $\int_1^2 \int_0^{2z} \int_0^{\ln x} xe^{-y}\, dy\, dx\, dz$

$$\int_1^2 \int_0^{2z} \int_0^{\ln x} xe^{-y}\, dy\, dx\, dz = \int_1^2 \int_0^{2z} x\left[-e^{-y}\right]_{y=0}^{\ln x} dx\, dz$$

$$= \int_1^2 \int_0^{2z} x\left[-e^{-\ln x} + 1\right] dx\, dz$$

$$= \int_1^2 \int_0^{2z} x\left[1 - \frac{1}{x}\right] dx\, dz$$

$$= \int_1^2 \int_0^{2z} (x-1)\, dx\, dz$$

$$= \int_1^2 \left[\frac{x^2}{2} - x\right]_{x=0}^{x=2z} d$$

$$= \int_1^2 \left[\left(\frac{4z^2}{2} - 2z\right) - (0-0)\right] dz$$

$$= \int_1^2 \left(2z^2 - 2z\right) dz$$

$$= \frac{2}{3}z^3 - z^2 \Big|_1^2$$

$$= \left(\frac{2}{3}(2)^3 - (2)^2\right) - \left(\frac{2}{3}(1^3) - (1)^2\right)$$

$$= \left(\frac{16}{3} - 4\right) - \left(\frac{2}{3} - 1\right)$$

$$= \frac{14}{3} - 3$$

$$= \frac{5}{3}.$$